Integration Tricks

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Most functions can be *differentiated* directly.
However, most functions cannot be *integrated* directly.

Any polynomial with only x^n terms (n ∈ ℝ) can be integrated using $\int x^n dx = \frac{1}{n+1}x^{n+1} + c$, $x \neq -1$

Also sin x, cos x, e^x, $\frac{1}{x}$, etc. can be integrated directly.

For other functions there are many "tricks" for integration.
For IB HL only five of these tricks are required:

- Integration by substitution
- Integration by reverse chain rule for *linear* inner functions
- Integration by parts (HL only)
- Integration using inverse trig functions (HL only)
- Integration using partial fractions (HL only)

Integration by Substitution

Substitution can be used in many different situations.

A) Substitution can be used to transform a function into a form which can be directly integrated.
Example: $u = x - 2$ to transform the following:

- $\frac{1}{x-2}$
- $\sin(x-2)$

With a bit of practice "Type A" substitutions can be done in your head without writing anything down.

B) Substitution is also used for functions which are the product or quotient of two functions, where one part of the function is the derivative of another part.

Examples: letting $u = x^3 + x$ will simplify:

- $\int \frac{3x^2+1}{x^3+x}dx$
- $\int (3x^2+1)\sin(x^3+x)dx$ (then integration by parts)

C) For functions which are the quotient of two polynomials, where the denominator is complicated, substitution can be used to move the complication to the numerator where it can be dealt with.

Example:

- $\int \frac{x^3}{(x-2)^2}dx$. Use $u = x - 2$ to transform it to

  $\int \frac{(u+2)^3}{u^2}du$; then expand $(u + 2)^3$.

Integration by reverse chain rule for *linear* inner functions

Integrals of the form $\int f(ax+b)dx$ can be easily integrated

Examples:

$$\int (ax+b)^n dx = \frac{1}{a}\times\frac{1}{n+1}(ax+b)^{n+1} + C$$

$$\int \cos(ax+b)dx = \frac{1}{a}\sin(ax+b) + C$$

$$\int \frac{1}{ax+b}dx = \frac{1}{a}\ln|ax+b| + C$$

Integration using Inverse Trig Functions

Inverse trig functions are used for integrals of the form:

- $\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + c$
- $\int \frac{1}{\sqrt{a^2-x^2}}dx = \arcsin\left(\frac{x}{a}\right) + c$

Sometimes one must first complete the square and make a (type A) substitution to get the function into one of the preceding forms.
Examples:

$$\int \frac{1}{x^2-6x+13}dx = \int \frac{1}{(x-3)^2+4} = \frac{1}{2}\arctan\left(\frac{x-3}{2}\right) + c$$

$$\int \frac{1}{\sqrt{-x^2-4x+5}}dx = \int \frac{1}{\sqrt{9-(x+2)^2}}dx = \arcsin\left(\frac{x+2}{3}\right) + c$$

Integration by Partial Fractions

Rational functions with a quadratic denominator cannot be directly integrated but can be broken up into to two rational functions with linear denominators, which can be integrated.

Example:

a) Show $\frac{x+7}{x^2-x-6} = \frac{A}{x-3} + \frac{B}{x+2}$, where A & B are constants to be determined.

$\frac{x+7}{x^2-x-6} = \frac{A(x+2)+B(x-3)}{x^2-x-6}$.
So $x + 7 = A(x + 2) + B(x - 3)$.
Letting $x = -2$ gives $5 = -5B$, so $B = -1$.
Letting $x = 3$ gives $10 = 5A$, so $A = 2$.

b) Integrate $\int \frac{x+7}{x^2-x-6}dx$

$= \int \frac{2}{x-3}dx - \int \frac{1}{x+2}dx$

$= 2\ln(x-3) - \ln(x+2) + C = \ln\left(\frac{(x-3)^2}{x+2}\right) + C$.