Vertical & Horizontal Asymptotes and Limits of Rational Functions

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An **asymptote** is a line that the graph of an equation approaches, but never reaches.

A **rational function** is a polynomial divided by another polynomial.

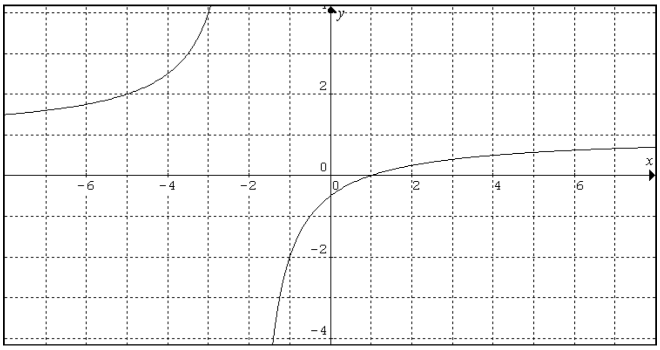
Vertical Asymptotes

A function approaches either +∞ or -∞ as x approaches a vertical asymptote.
There is a vertical asymptote at those values of x that make the denominator equal to zero.

Example $f(x) = \frac{x-1}{x+2}$.

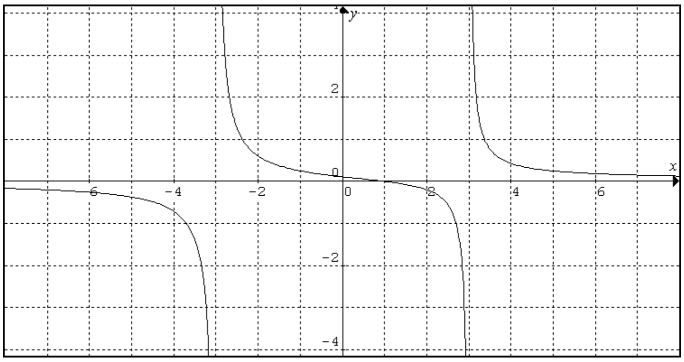
There is a vertical asymptote if x + 2 = 0.
Therefore the line x = -2 is a vertical asymptote.

Therefore $\lim_{x \to -2} f(x) = \pm\infty$



Example $f(x) = \frac{x-1}{x^2-9}$.

There is a vertical asymptote if $x^2 - 9 = 0$.
So there is a vertical asymptote if
(x - 3)(x + 3) = 0.
Therefore the lines x = 3 and x = -3 are vertical asymptotes.



The graph of a function **cannot** cross a vertical asymptote, because if it did, then it would not be a function.

Horizontal Asymptotes

A horizontal asymptote is a horizontal line that the function approaches as x approaches +∞ or -∞.
A horizontal asymptote is the limit of y = f(x) as x approaches +∞ or -∞, i.e. $y = \lim_{x \to \infty} f(x)$.

A rational function is of the form

$$f(x) = \frac{a_n x^n + \ldots + a_1 x^1 + a_0}{b_m x^m + \ldots + b_1 x^1 + b_0}$$

To investigate horizontal asymptotes, the only thing you need to consider is the relative values of **n & m,** the highest powers in the numerator and denominator respectively.

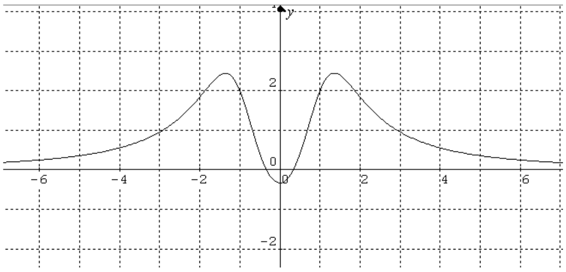
There are three cases:

1. If the power in the denominator is bigger, then the x-axis (y = 0) is a horizontal asymptote.

Example $f(x) = \frac{9x^2 - 1}{x^4 + 3}$. HA: $y = \lim_{x \to \infty} f(x) = 0$

Since 2 < 4, the line y = 0 is a horizontal asymptote.

Therefore $\lim_{x \to \pm\infty} f(x) = 0$

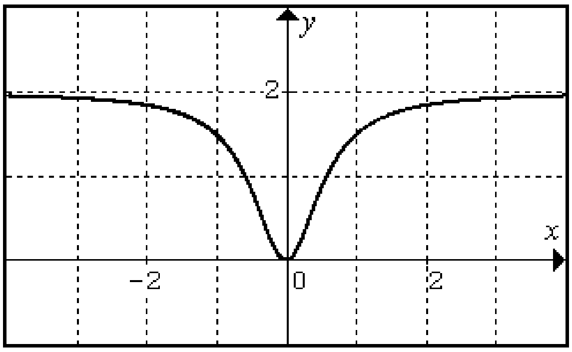


2. If the powers are the same, then there is a horizontal asymptote at y equals the ratios of the coefficients of the highest power terms in the numerator and denominator.

Example $f(x) = \frac{6x^2}{3x^2 + 1}$. HA: $y = \lim_{x \to \infty} f(x) = 2$

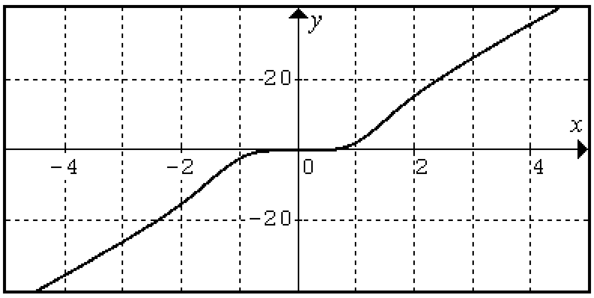
Since 2 = 2, the line $y = \frac{6}{3}$, i.e. y = 2 is a horizontal asymptote.

Therefore $\lim_{x \to \pm\infty} f(x) = 2$



3. If the power in the numerator is bigger, there is no horizontal asymptote.

Example $f(x) = \frac{9x^5 - 1}{x^4 + 3}$.



The graph of a function **can** cross a horizontal asymptote. See, for example, the graph of $f(x) = \frac{9x^2 - 1}{x^4 + 3}$ above.