Mathematical Induction 2008-14 with MS

1a. *[4 marks]*

Using the definition of a derivative as $f'(x) = \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$, show that the derivative of $\frac{1}{2x+1}$ is $\frac{-2}{(2x+1)^2}$.

1b. *[9 marks]*

Prove by induction that the n^{th} derivative of $(2x+1)^{-1}$ is $(-1)^n \frac{2^n n!}{(2x+1)^{n+1}}$.

2a. *[3 marks]*

Consider a function f, defined by $f(x) = \frac{x}{2-x}$ for $0 \leqslant x \leqslant 1$.

Find an expression for $(f \circ f)(x)$.

2b. *[8 marks]*

Let $F_n(x) = \frac{x}{2^n - (2^n - 1)x}$, where $0 \leqslant x \leqslant 1$.

Use mathematical induction to show that for any $n \in \mathbb{Z}^+$

$$\underbrace{(f \circ f \circ \ldots \circ f)}_{n \text{ times}}(x) = F_n(x)$$

.

2c. *[6 marks]*

Show that $F_{-n}(x)$ is an expression for the inverse of F_n.

2d. *[6 marks]*

(i) State $F_n(0)$ and $F_n(1)$.

(ii) Show that $F_n(x) < x$, given $0 < x < 1$, $n \in \mathbb{Z}^+$.

(iii) For $n \in \mathbb{Z}^+$, let A_n be the area of the region enclosed by the graph of F_n^{-1}, the x-axis and the line $x = 1$. Find the area B_n of the region enclosed by F_n and F_n^{-1} in terms of A_n.

3a. *[3 marks]*

Find the sum of the infinite geometric sequence 27, −9, 3, −1, ...

3b. *[7 marks]*

Use mathematical induction to prove that for $n \in \mathbb{Z}^+$,

$$a + ar + ar^2 + \ldots + ar^{n-1} = \frac{a(1-r^n)}{1-r}.$$

4a. *[8 marks]*

Prove by mathematical induction that, for $n \in \mathbb{Z}^+$,

$$1 + 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + 4\left(\frac{1}{2}\right)^3 + \ldots + n\left(\frac{1}{2}\right)^{n-1} = 4 - \frac{n+2}{2^{n-1}}.$$

4b. *[17 marks]*

(a) Using integration by parts, show that $\int e^{2x} \sin x \, dx = \frac{1}{5} e^{2x}(2\sin x - \cos x) + C$.

(b) Solve the differential equation $\frac{dy}{dx} = \sqrt{1-y^2} e^{2x} \sin x$, given that $y = 0$ when $x = 0$,

writing your answer in the form $y = f(x)$.

(c) (i) Sketch the graph of $y = f(x)$, found in part (b), for $0 \leqslant x \leqslant 1.5$.

Determine the coordinates of the point P, the first positive intercept on the *x*-axis, and mark it on your sketch.

(ii) The region bounded by the graph of $y = f(x)$ and the *x*-axis, between the origin and P, is rotated 360° about the *x*-axis to form a solid of revolution.

Calculate the volume of this solid.

5. *[8 marks]*

Prove by mathematical induction $\sum_{r=1}^{n} r(r!) = (n+1)! - 1$, $n \in \mathbb{Z}^+$.

6a. *[14 marks]*

(a) The sum of the first six terms of an arithmetic series is 81. The sum of its first eleven terms is 231. Find the first term and the common difference.

(b) The sum of the first two terms of a geometric series is 1 and the sum of its first four terms is 5. If all of its terms are positive, find the first term and the common ratio.

(c) The r^{th} term of a new series is defined as the product of the r^{th} term of the arithmetic series and the r^{th} term of the geometric series above. Show that the r^{th} term of this new series is $(r+1)2^{r-1}$.

6b. *[7 marks]*

Using mathematical induction, prove that

$$\sum_{r=1}^{n} (r+1)2^{r-1} = n2^n, \; n \in \mathbb{Z}^+.$$

7a. *[3 marks]*

The function *f* is defined by $f(x) = \mathrm{e}^x \sin x$.

Show that $f''(x) = 2\mathrm{e}^x \sin\left(x + \frac{\pi}{2}\right)$.

7b. *[4 marks]*

Obtain a similar expression for $f^{(4)}(x)$.

7c. *[8 marks]*

Suggest an expression for $f^{(2n)}(x)$, $n \in \mathbb{Z}^+$, and prove your conjecture using mathematical induction.

8. *[20 marks]*

(a) Show that $\sin 2nx = \sin((2n+1)x)\cos x - \cos((2n+1)x)\sin x$.

(b) **Hence** prove, by induction, that

$$\cos x + \cos 3x + \cos 5x + \ldots + \cos((2n-1)x) = \frac{\sin 2nx}{2\sin x},$$

for all $n \in \mathbb{Z}^+$, $\sin x \neq 0$.

(c) Solve the equation $\cos x + \cos 3x = \frac{1}{2}$, $0 < x < \pi$.

9. *[10 marks]*

(a) Consider the following sequence of equations.

$1 \times 2 = \frac{1}{3}(1 \times 2 \times 3),$

$1 \times 2 + 2 \times 3 = \frac{1}{3}(2 \times 3 \times 4),$

$1 \times 2 + 2 \times 3 + 3 \times 4 = \frac{1}{3}(3 \times 4 \times 5),$

$\ldots.$

(i) Formulate a conjecture for the n^{th} equation in the sequence.

(ii) Verify your conjecture for $n = 4$.

(b) A sequence of numbers has the n^{th} term given by $u_n = 2^n + 3,\ n \in \mathbb{Z}^+$. Bill conjectures that all members of the sequence are prime numbers. Show that Bill's conjecture is false.

(c) Use mathematical induction to prove that $5 \times 7^n + 1$ is divisible by 6 for all $n \in \mathbb{Z}^+$.

10. *[7 marks]*

Use the method of mathematical induction to prove that $5^{2n} - 24n - 1$ is divisible by 576 for $n \in \mathbb{Z}^+$.

11. *[7 marks]*

Given that $y = \frac{1}{1-x}$, use mathematical induction to prove that $\frac{\mathrm{d}^n y}{\mathrm{d}x^n} = \frac{n!}{(1-x)^{n+1}}, \ n \in \mathbb{Z}^+$.

12. *[8 marks]*

Prove, by mathematical induction, that $7^{8n+3} + 2, \ n \in \mathbb{N}$, is divisible by 5.

13a. *[7 marks]*

Consider $z = r(\cos\theta + \mathrm{i}\sin\theta), \ z \in \mathbb{C}$.

Use mathematical induction to prove that $z^n = r^n(\cos n\theta + \mathrm{i}\sin n\theta), \ n \in \mathbb{Z}^+$.

13b. *[4 marks]*

Given $u = 1 + \sqrt{3}\mathrm{i}$ and $v = 1 - \mathrm{i}$,

(i) express u and v in modulus-argument form;

(ii) hence find $u^3 v^4$.

13c. *[1 mark]*

The complex numbers$u$ and v are represented by point A and point B respectively on an Argand diagram.

Plot point A and point B on the Argand diagram.

13d. *[3 marks]*

Point A is rotated through $\frac{\pi}{2}$ in the anticlockwise direction about the origin O to become point A'.
Point B is rotated through $\frac{\pi}{2}$ in the clockwise direction about O to become point B'.

Find the area of triangle O$\mathrm{A}'\mathrm{B}'$.

13e. *[5 marks]*

Given that u and v are roots of the equation $z^4 + bz^3 + cz^2 + dz + e = 0$, where $b, \ c, \ d, \ e \in \mathbb{R}$,

find the values of $b, \ c, \ d$ and e.

14. *[7 marks]*

Prove by mathematical induction that $n^3 + 11n$ is divisible by 3 for all $n \in \mathbb{Z}^+$.

Mathematical Induction 2008-14 MS

1a. *[4 marks]*

Markscheme

let $f(x) = \frac{1}{2x+1}$ and using the result $f'(x) = \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$

$f'(x) = \lim_{h\to 0}\left(\frac{\frac{1}{2(x+h)+1}-\frac{1}{2x+1}}{h}\right)$ ***M1A1***

$\Rightarrow f'(x) = \lim_{h\to 0}\left(\frac{[2x+1]-[2(x+h)+1]}{h[2(x+h)+1][2x+1]}\right)$ ***A1***

$\Rightarrow f'(x) = \lim_{h\to 0}\left(\frac{-2}{[2(x+h)+1][2x+1]}\right)$ ***A1***

$\Rightarrow f'(x) = \frac{-2}{(2x+1)^2}$ ***AG***

[4 marks]

Examiners report

Even though the definition of the derivative was given in the question, solutions to (a) were often disappointing with algebraic errors fairly common, usually due to brackets being omitted or manipulated incorrectly. Solutions to the proof by induction in (b) were often poor. Many candidates fail to understand that they have to assume that the result is true for $n = k$ and then show that this leads to it being true for $n = k + 1$. Many candidates just write 'Let $n = k$' which is of course meaningless. The conclusion is often of the form 'True for $n = 1,\ n = k$ and $n = k + 1$ therefore true by induction'. Credit is only given for a conclusion which includes a statement such as 'True for $n = k \Rightarrow$ true for $n = k + 1$'.

1b. *[9 marks]*

Markscheme

let $y = \frac{1}{2x+1}$

we want to prove that $\frac{\mathrm{d}^n y}{\mathrm{d}x^n} = (-1)^n \frac{2^n n!}{(2x+1)^{n+1}}$

let $n = 1 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = (-1)^1 \frac{2^1 1!}{(2x+1)^{1+1}}$ ***M1***

$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-2}{(2x+1)^2}$ which is the same result as part (a)

hence the result is true for $n = 1$ ***R1*** assume the result is true for $n = k$: $\frac{\mathrm{d}^k y}{\mathrm{d}x^k} = (-1)^k \frac{2^k k!}{(2x+1)^{k+1}}$ ***M1***

$\frac{\mathrm{d}^{k+1} y}{\mathrm{d}x^{k+1}} = \frac{\mathrm{d}}{\mathrm{d}x}\left[(-1)^k \frac{2^k k!}{(2x+1)^{k+1}}\right]$ ***M1***

$\Rightarrow \frac{\mathrm{d}^{k+1} y}{\mathrm{d}x^{k+1}} = \frac{\mathrm{d}}{\mathrm{d}x}\left[(-1)^k 2^k k!(2x+1)^{-k-1}\right]$ ***(A1)***

$\Rightarrow \frac{\mathrm{d}^{k+1} y}{\mathrm{d}x^{k+1}} = (-1)^k 2^k k!(-k-1)(2x+1)^{-k-2} \times 2$ ***A1***

$\Rightarrow \frac{\mathrm{d}^{k+1} y}{\mathrm{d}x^{k+1}} = (-1)^{k+1} 2^{k+1} (k+1)!(2x+1)^{-k-2}$ ***(A1)***

$\Rightarrow \frac{\mathrm{d}^{k+1} y}{\mathrm{d}x^{k+1}} = (-1)^{k+1} \frac{2^{k+1}(k+1)!}{(2x+1)^{k+2}}$ ***A1***

hence if the result is true for $n = k$, it is true for $n = k + 1$

since the result is true for $n = 1$, the result is proved by mathematical induction ***R1***

Note: Only award final ***R1*** if all the ***M*** marks have been gained.

[9 marks]

Examiners report

Even though the definition of the derivative was given in the question, solutions to (a) were often disappointing with algebraic errors fairly common, usually due to brackets being omitted or manipulated incorrectly. Solutions to the proof by induction in (b) were often poor. Many candidates fail to understand that they have to assume that the result is true for $n = k$ and then show that this leads to it being true for $n = k + 1$. Many candidates just write 'Let $n = k$' which is of course meaningless. The conclusion is often of the form 'True for $n = 1,\ n = k$ and $n = k + 1$ therefore true by induction'. Credit is only given for a conclusion which includes a statement such as 'True for $n = k \Rightarrow$ true for $n = k + 1$'.

2a. *[3 marks]*

Markscheme

$(f \circ f)(x) = f\left(\frac{x}{2-x}\right) = \frac{\frac{x}{2-x}}{2-\frac{x}{2-x}}$ ***M1A1***

$(f \circ f)(x) = \frac{x}{4-3x}$ ***A1***

[3 marks]

Examiners report

Part a) proved to be an easy 3 marks for most candidates.

2b. *[8 marks]*

Markscheme

$$P(n): \underbrace{(f \circ f \circ \ldots \circ f)}_{n \text{ times}}(x) = F_n(x)$$

$P(1):\ f(x) = F_1(x)$

$LHS = f(x) = \frac{x}{2-x}$ and $RHS = F_1(x) = \frac{x}{2^1 - (2^1 - 1)x} = \frac{x}{2-x}$ ***A1A1***

$\therefore P(1)$ true

assume that $P(k)$ is true, *i.e.*, $\underbrace{(f \circ f \circ \ldots \circ f)}_{\text{k times}}(x) = F_k(x)$ ***M1***

consider $P(k+1)$

EITHER

$$\underbrace{(f \circ f \circ \ldots \circ f)}_{\text{k+1 times}}(x) = \left(f \circ \underbrace{f \circ f \circ \ldots \circ f}_{\text{k times}}\right)(x) = f\left(F_k(x)\right)$$ ***(M1)***

$= f\left(\frac{x}{2^k - (2^k - 1)x}\right) = \frac{\frac{x}{2^k - (2^k - 1)x}}{2 - \frac{x}{2^k - (2^k - 1)x}}$ ***A1***

$= \frac{x}{2(2^k - (2^k - 1)x) - x} = \frac{x}{2^{k+1} - (2^{k+1} - 2)x - x}$ ***A1***

OR

$$\underbrace{(f \circ f \circ \ldots \circ f)}_{\text{k+1 times}}(x) = \left(f \circ \underbrace{f \circ f \circ \ldots \circ f}_{\text{k times}}\right)(x) = F_k(f(x))$$ ***(M1)***

$= F_k\left(\frac{x}{2-x}\right) = \frac{\frac{x}{2-x}}{2^k - (2^k - 1)\frac{x}{2-x}}$ ***A1***

$= \frac{x}{2^{k+1} - 2^k x - 2^k x + x}$ ***A1***

THEN

$= \frac{x}{2^{k+1} - (2^{k+1} - 1)x} = F_{k+1}(x)$ ***A1***

$P(k)$ true implies $P(k+1)$ true, $P(1)$ true so $P(n)$ true for all $n \in \mathbb{Z}^+$ ***R1***

[8 marks]

Examiners report

Part b) was often answered well, and candidates were well prepared in this session for this type of question. Candidates still need to take care when showing explicitly that P(1) is true, and some are still writing ‘Let $n = k$’ which gains no marks. The inductive step was often well argued, and given in clear detail, though the final inductive reasoning step was incorrect, or appeared rushed, even from the better candidates. ‘True for $n = 1$, $n = k$ and $n = k + 1$’ is still disappointingly seen, as were some even more unconvincing variations.

2c. *[6 marks]*

Markscheme

METHOD 1

$x = \frac{y}{2^n - (2^n - 1)y} \Rightarrow 2^n x - (2^n - 1)xy = y$ ***M1A1***

$\Rightarrow 2^n x = ((2^n - 1)x + 1)\, y \Rightarrow y = \frac{2^n x}{(2^n - 1)x + 1}$ ***A1***

$F_n^{-1}(x) = \frac{2^n x}{(2^n - 1)x + 1}$ ***A1***

$F_n^{-1}(x) = \frac{x}{\frac{2^n - 1}{2^n} x + \frac{1}{2^n}}$ ***M1***

$F_n^{-1}(x) = \frac{x}{(1 - 2^{-n})x + 2^{-n}}$ ***A1***

$F_n^{-1}(x) = \frac{x}{2^{-n} - (2^{-n} - 1)x}$ ***AG***

METHOD 2

attempt $F_{-n}\left(F_n(x)\right)$ ***M1***

$= F_{-n}\left(\frac{x}{2^n - (2^n - 1)x}\right) = \frac{\frac{x}{2^n - (2^n - 1)x}}{2^{-n} - (2^{-n} - 1)\frac{x}{2^n - (2^n - 1)x}}$ ***A1A1***

$= \frac{x}{2^{-n}(2^n - (2^n - 1)x) - (2^{-n} - 1)x}$ ***A1A1***

Note: Award ***A1*** marks for numerators and denominators.

$= \frac{x}{1} = x$ ***A1AG***

METHOD 3

attempt $F_n\left(F_{-n}(x)\right)$ **M1**

$= F_n\left(\frac{x}{2^{-n} - (2^{-n} - 1)x}\right) = \frac{\frac{x}{2^{-n} - (2^{-n} - 1)x}}{2^n - (2^n - 1)\frac{x}{2^{-n} - (2^{-n} - 1)x}}$ ***A1A1***

$= \frac{x}{2^n(2^{-n} - (2^{-n} - 1)x) - (2^n - 1)x}$ ***A1A1***

Note: Award ***A1*** marks for numerators and denominators.

$= \frac{x}{1} = x$ ***A1AG***

[6 marks]

Examiners report

Part c) was again very well answered by the majority. A few weaker candidates attempted to find an inverse for the individual case n = 1 , but gained no credit for this.

2d. *[6 marks]*

Markscheme

(i) $F_n(0) = 0,\ F_n(1) = 1$ ***A1***

(ii) **METHOD 1**

$2^n - (2^n - 1)x - 1 = (2^n - 1)(1 - x)$ ***(M1)***

> 0 if $0 < x < 1$ and $n \in \mathbb{Z}^+$ ***A1***

so $2^n - (2^n - 1)x > 1$ and $F_n(x) = \frac{x}{2^n - (2^n - 1)x} < \frac{x}{1}\ (< x)$ ***R1***

$F_n(x) = \frac{x}{2^n - (2^n - 1)x} < x$ for $0 < x < 1$ and $n \in \mathbb{Z}^+$ ***AG***

METHOD 2

$\frac{x}{2^n - (2^n - 1)x} < x \Leftrightarrow 2^n - (2^n - 1)x > 1$ ***(M1)***

$\Leftrightarrow (2^n - 1)x < 2^n - 1$ ***A1***

$\Leftrightarrow x < \frac{2^n - 1}{2^n - 1} = 1$ true in the interval $]0,\ 1[$ ***R1***

(iii) $B_n = 2\left(A_n - \frac{1}{2}\right)\ (= 2A_n - 1)$ ***(M1)A1***

[6 marks]

Examiners report

Part d) was not at all well understood, with virtually no candidates able to tie together the hints given by connecting the different parts of the question. Rash, and often thoughtless attempts were made at each part, though by this stage some seemed to be struggling through lack of time. The inequality part of the question tended to be ‘fudged’, with arguments seen by examiners being largely unconvincing and lacking clarity. A tiny number of candidates provided the correct answer to the final part, though a surprising number persisted with what should have been recognised as fruitless working – usually in the form of long-winded integration attempts.

3a. *[3 marks]*

Markscheme

$r = -\frac{1}{3}$ ***(A1)***

$S_\infty = \frac{27}{1+\frac{1}{3}}$ ***M1***

$S_\infty = \frac{81}{4} \quad (= 20.25)$ ***A1 N1***

[3 marks]

Examiners report

Part (a) was correctly answered by the majority of candidates, although a few found $r = -3$.

3b. *[7 marks]*

Markscheme

Attempting to show that the result is true for $n = 1$ ***M1***

LHS = a **and** $\text{RHS} = \frac{a(1-r)}{1-r} = a$ ***A1***

Hence the result is true for $n = 1$

Assume it is true for $n = k$

$a + ar + ar^2 + \ldots + ar^{k-1} = \frac{a(1-rk)}{1-r}$ ***M1***

Consider $n = k + 1$:

$a + ar + ar^2 + \ldots + ar^{k-1} + ar^k = \frac{a(1-r^k)}{1-r} + ar^k$ ***M1***

$= \frac{a(1-r^k)+ar^k(1-r)}{1-r}$

$= \frac{a-ar^k+ar^k-ar^{k+1}}{1-r}$ ***A1***

Note: Award ***A1*** for an equivalent correct intermediate step.

$= \frac{a-ar^{k+1}}{1-r}$

$= \frac{a(1-r^{k+1})}{1-r}$ ***A1***

Note: Illogical attempted proofs that use the result to be proved would gain ***M1A0A0*** for the last three above marks.

The result is true for $n = k \Rightarrow$ it is true for $n = k + 1$ **and** as it is true for $n = 1$, the result is proved by mathematical induction. ***R1 N0***

Note: To obtain the final ***R1*** mark a reasonable attempt must have been made to prove the $k + 1$ step.

[7 marks]

Examiners report

Part (b) was often started off well, but a number of candidates failed to initiate the $n = k + 1$ step in a satisfactory way. A number of candidates omitted the 'P(1) is true' part of the concluding statement.

4a. *[8 marks]*

Markscheme

prove that $1+2\left(\frac{1}{2}\right)+3\left(\frac{1}{2}\right)^2+4\left(\frac{1}{2}\right)^3+\ldots+n\left(\frac{1}{2}\right)^{n-1}=4-\frac{n+2}{2^{n-1}}$

for $n = 1$

$\text{LHS} = 1,\ \text{RHS} = 4 - \frac{1+2}{2^0} = 4 - 3 = 1$

so true for $n = 1$ ***R1***

assume true for $n = k$ ***M1***

so $1+2\left(\frac{1}{2}\right)+3\left(\frac{1}{2}\right)^2+4\left(\frac{1}{2}\right)^3+\ldots+k\left(\frac{1}{2}\right)^{k-1}=4-\frac{k+2}{2^{k-1}}$

now for $n = k + 1$

LHS: $1+2\left(\frac{1}{2}\right)+3\left(\frac{1}{2}\right)^2+4\left(\frac{1}{2}\right)^3+\ldots+k\left(\frac{1}{2}\right)^{k-1}+(k+1)\left(\frac{1}{2}\right)^k$ ***A1***

$=4-\frac{k+2}{2^{k-1}}+(k+1)\left(\frac{1}{2}\right)^k$ ***M1A1***

$=4-\frac{2(k+2)}{2^k}+\frac{k+1}{2^k}$ (or equivalent) ***A1***

$=4-\frac{(k+1)+2}{2^{(k+1)-1}}$ (accept $4-\frac{k+3}{2^k}$) ***A1***

Therefore if it is true for $n = k$ it is true for $n = k + 1$. It has been shown to be true for $n = 1$ so it is true for all $n\ (\in \mathbb{Z}^+)$. ***R1***

Note: To obtain the final ***R*** mark, a reasonable attempt at induction must have been made.

[8 marks]

Examiners report

Part A: Given that this question is at the easier end of the 'proof by induction' spectrum, it was disappointing that so many candidates failed to score full marks. The $n = 1$ case was generally well done. The whole point of the method is that it involves logic, so 'let n = k' or 'put n = k', instead of 'assume ... to be true for n = k', gains no marks. The algebraic steps need to be more convincing than some candidates were able to show. It is astonishing that the R1 mark for the final statement was so often not awarded.

4b. *[17 marks]*

Markscheme

(a)

METHOD 1

$\int e^{2x} \sin x dx = -\cos x e^{2x} + \int 2e^{2x} \cos x dx$ ***M1A1A1***

$= -\cos x e^{2x} + 2e^{2x} \sin x - \int 4e^{2x} \sin x dx$ ***A1A1***

$5 \int e^{2x} \sin x dx = -\cos x e^{2x} + 2e^{2x} \sin x$ ***M1***

$\int e^{2x} \sin x dx = \frac{1}{5} e^{2x} (2 \sin x - \cos x) + C$ ***AG***

METHOD 2

$\int \sin x e^{2x} dx = \frac{\sin x e^{2x}}{2} - \int \cos x \frac{e^{2x}}{2} dx$ ***M1A1A1***

$= \frac{\sin x e^{2x}}{2} - \cos x \frac{e^{2x}}{4} - \int \sin x \frac{e^{2x}}{4} dx$ ***A1A1***

$\frac{5}{4} \int e^{2x} \sin x dx = \frac{e^{2x} \sin x}{2} - \frac{\cos x e^{2x}}{4}$ ***M1***

$\int e^{2x} \sin x dx = \frac{1}{5} e^{2x} (2 \sin x - \cos x) + C$ ***AG***

[6 marks]

(b)

$\int \frac{dy}{\sqrt{1-y^2}} = \int e^{2x} \sin x dx$ ***M1A1***

$\arcsin y = \frac{1}{5} e^{2x} (2 \sin x - \cos x)(+C)$ ***A1***

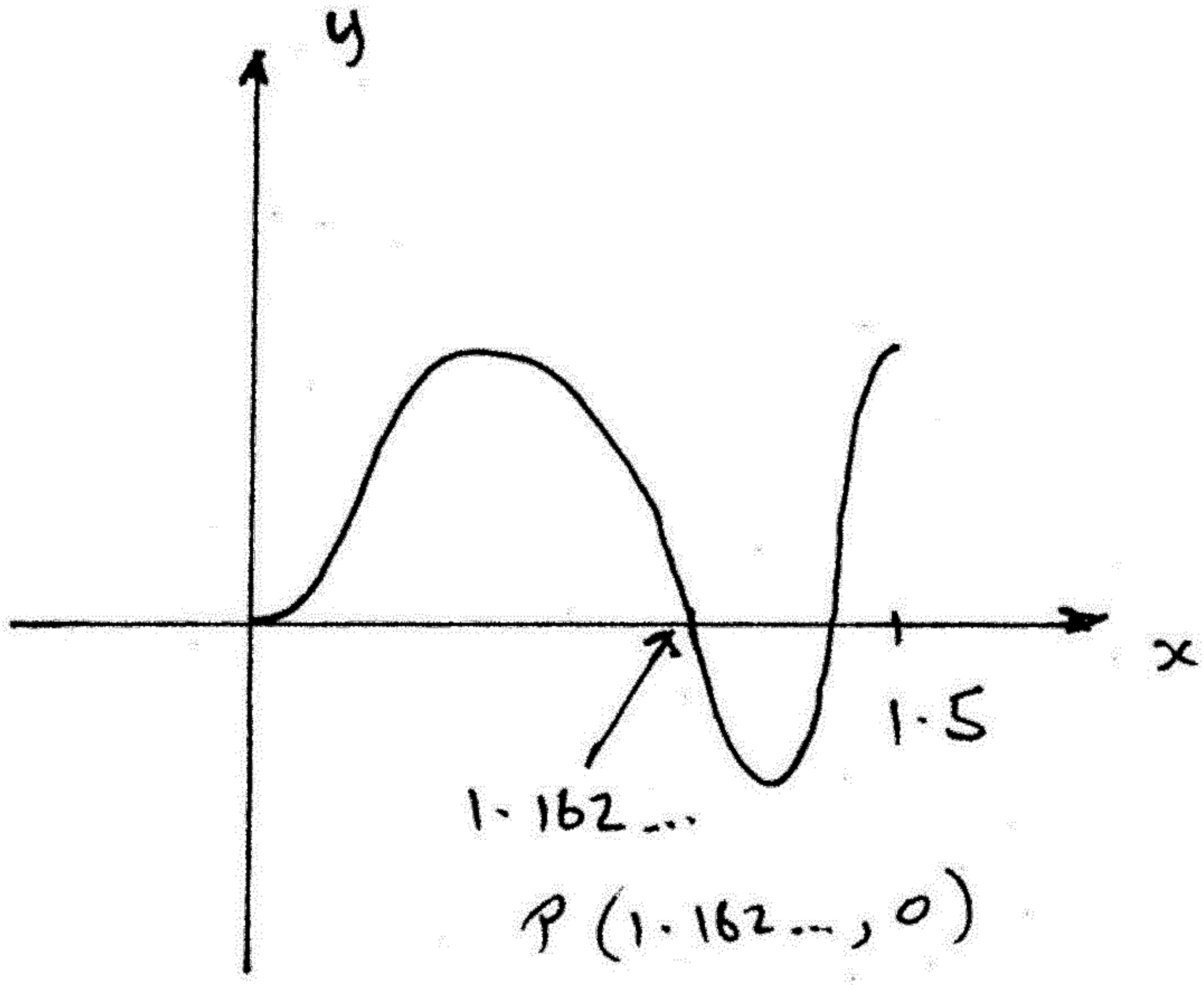
when $x = 0,\ y = 0 \Rightarrow C = \frac{1}{5}$ ***M1***

$y = \sin \left(\frac{1}{5} e^{2x} (2 \sin x - \cos x) + \frac{1}{5}\right)$ ***A1***

[5 marks]

(c)

(i)



A1

P is (1.16, 0) ***A1***

Note: Award ***A1*** for 1.16 seen anywhere, ***A1*** for complete sketch.

Note: Allow FT on their answer from (b)

(ii) $V = \int_0^{1.162...} \pi y^2 \mathrm{d}x$ ***M1A1***

$= 1.05$ ***A2***

Note: Allow FT on their answers from (b) and (c)(i).

[6 marks]

Examiners report

Part B: Part (a) was often well done, although some faltered after the first integration. Part (b) was also generally well done, although there were some errors with the constant of integration. In (c) the graph was often attempted, but errors in (b) usually led to manifestly incorrect plots. Many attempted the volume of integration and some obtained the correct value.

5. *[8 marks]*

Markscheme

let $n = 1$

LHS $= 1 \times 1! = 1$

RHS $= (1 + 1)! - 1 = 2 - 1 = 1$

hence true for $n = 1$ ***R1***

assume true for $n = k$

$\sum\limits_{r=1}^{k} r(r!) = (k + 1)! - 1$ ***M1***

$\sum\limits_{r=1}^{k+1} r(r!) = (k + 1)! - 1 + (k + 1) \times (k + 1)!$ ***M1A1***

$= (k + 1)!(1 + k + 1) - 1$

$= (k + 1)!(k + 2) - 1$ ***A1***

$= (k + 2)! - 1$ ***A1***

hence if true for $n = k$, true for $n = k + 1$ ***R1***

since the result is true for $n = 1$ and $\mathrm{P}(k) \Rightarrow \mathrm{P}(k + 1)$ the result is proved by mathematical induction $\forall n \in \mathbb{Z}^{+}$ ***R1***

[8 marks]

Examiners report

This question was done poorly on a number of levels. Many students knew the structure of induction but did not show that they understood what they were doing. The general notation was poor for both the induction itself and the sigma notation.

In noting the case for $n = 1$ too many stated the equation rather than using the LHS and RHS separately and concluding with a statement. There were also too many who did not state the conclusion for this case.

Many did not state the assumption for $n = k$ as an assumption.

Most stated the equation for $n = k + 1$ and worked with the equation. Also common was the lack of sigma and inappropriate use of *n* and *k* in the statement. There were some very nice solutions however.

The final conclusion was often not complete or not considered which would lead to the conclusion that the student did not really understand what induction is about.

6a. *[14 marks]*

Markscheme

(a) $S_6 = 81 \Rightarrow 81 = \frac{6}{2}(2a + 5d)$ ***M1A1***

$\Rightarrow 27 = 2a + 5d$

$S_{11} = 231 \Rightarrow 231 = \frac{11}{2}(2a + 10d)$ ***M1A1***

$\Rightarrow 21 = a + 5d$

solving simultaneously, *a* = 6 , *d* = 3 ***A1A1***

[6 marks]

(b) $a + ar = 1$ ***A1***

$a + ar + ar^2 + ar^3 = 5$ ***A1***

$\Rightarrow (a + ar) + ar^2(1 + r) = 5$

$\Rightarrow 1 + ar^2 \times \frac{1}{a} = 5$

obtaining $r^2 - 4 = 0$ ***M1***

$\Rightarrow r = \pm 2$

$r = 2$ (since all terms are positive) ***A1***

$a = \frac{1}{3}$ ***A1***

[5 marks]

(c) AP r^{th} term is $3r + 3$ ***A1***

GP r^{th} term is $\frac{1}{3}2^{r-1}$ ***A1***

$3(r + 1) \times \frac{1}{3}2^{r-1} = (r + 1)2^{r-1}$ ***M1AG***

[3 marks]

Total [14 marks]

Examiners report

Parts (a), (b) and (c) were answered successfully by a large number of candidates. Some, however, had difficulty with the arithmetic.

6b. *[7 marks]*

Markscheme

prove: $P_n : \sum_{r=1}^{n}(r + 1)2^{r-1} = n2^n,\ n \in \mathbb{Z}^+.$

show true for *n* = 1 , *i.e.*

$\text{LHS} = 2 \times 2^0 = 2 = \text{RHS}$ ***A1***

assume true for $n = k$, *i.e.* ***M1***

$\sum_{r=1}^{k} (r+1)2^{r-1} = k2^k,\ k \in \mathbb{Z}^+$

consider $n = k + 1$

$\sum_{r=1}^{k+1} (r+1)2^{r-1} = k2^k + (k+2)2^k$ ***M1A1***

$= 2^k(k + k + 2)$

$= 2(k+1)2^k$ ***A1***

$= (k+1)2^{k+1}$ ***A1***

hence true for $n = k + 1$

P_{k+1} is true whenever P_k is true, and P_1 is true, therefore P_n is true ***R1***

for $n \in \mathbb{Z}^+$

[7 marks]

Examiners report

In part (d) many candidates showed little understanding of sigma notation and proof by induction. There were cases of circular reasoning and using *n*, *k* and *r* randomly. A concluding sentence almost always appeared, even if the proof was done incorrectly, or not done at all.

7a. *[3 marks]*

Markscheme

$f'(x) = \mathrm{e}^x \sin x + \mathrm{e}^x \cos x$ ***A1***

$f''(x) = \mathrm{e}^x \sin x + \mathrm{e}^x \cos x + \mathrm{e}^x \cos x - \mathrm{e}^x \sin x$ ***A1***

$= 2\mathrm{e}^x \cos x$ ***A1***

$= 2\mathrm{e}^x \sin\left(x + \frac{\pi}{2}\right)$ ***AG***

[3 marks]

Examiners report

[N/A]

7b. *[4 marks]*

Markscheme

$f'''(x) = 2e^x \sin\left(x + \frac{\pi}{2}\right) + 2e^x \cos\left(x + \frac{\pi}{2}\right)$ ***A1***

$f^{(4)}(x) = 2e^x \sin\left(x + \frac{\pi}{2}\right) + 2e^x \cos\left(x + \frac{\pi}{2}\right) + 2e^x \cos\left(x + \frac{\pi}{2}\right) - 2e^x \sin\left(x + \frac{\pi}{2}\right)$ ***A1***

$= 4e^x \cos\left(x + \frac{\pi}{2}\right)$ ***A1***

$= 4e^x \sin(x + \pi)$ ***A1***

[4 marks]

Examiners report

[N/A]

7c. *[8 marks]*

Markscheme

the conjecture is that

$f^{(2n)}(x) = 2^n e^x \sin\left(x + \frac{n\pi}{2}\right)$ ***A1***

for *n* = 1, this formula gives

$f''(x) = 2e^x \sin\left(x + \frac{\pi}{2}\right)$ which is correct ***A1***

let the result be true for *n* = *k* , $\left(i.e.\ \ f^{(2k)}(x) = 2^k e^x \sin\left(x + \frac{k\pi}{2}\right)\right)$ ***M1***

consider $f^{(2k+1)}(x) = 2^k e^x \sin\left(x + \frac{k\pi}{2}\right) + 2^k e^x \cos\left(x + \frac{k\pi}{2}\right)$ ***M1***

$f^{(2(k+1))}(x) = 2^k e^x \sin\left(x + \frac{k\pi}{2}\right) + 2^k e^x \cos\left(x + \frac{k\pi}{2}\right) + 2^k e^x \cos\left(x + \frac{k\pi}{2}\right) - 2^k e^x \sin\left(x + \frac{k\pi}{2}\right)$ ***A1***

$= 2^{k+1} e^x \cos\left(x + \frac{k\pi}{2}\right)$ ***A1***

$= 2^{k+1} e^x \sin\left(x + \frac{(k+1)\pi}{2}\right)$ ***A1***

therefore true for $n = k \Rightarrow$ true for $n = k + 1$ and since true for $n = 1$

the result is proved by induction. ***R1***

Note: Award the final ***R1*** only if the two ***M*** marks have been awarded.

[8 marks]

Examiners report

[N/A]

8. *[20 marks]*

Markscheme

(a) $\sin(2n+1)x\cos x - \cos(2n+1)x\sin x = \sin(2n+1)x - x$ ***M1A1***

$= \sin 2nx$ ***AG***

[2 marks]

(b) if n = 1 ***M1***

$\text{LHS} = \cos x$

$\text{RHS} = \frac{\sin 2x}{2\sin x} = \frac{2\sin x\cos x}{2\sin x} = \cos x$ ***M1***

so LHS = RHS and the statement is true for n = 1 ***R1***

assume true for n = k ***M1***

Note: Only award ***M1*** if the word **true** appears.

Do **not** award ***M1*** for 'let $n = k$' only.

Subsequent marks are independent of this ***M1***.

so $\cos x + \cos 3x + \cos 5x + \ldots + \cos(2k-1)x = \frac{\sin 2kx}{2\sin x}$

if n = k + 1 then

$\cos x + \cos 3x + \cos 5x + \ldots + \cos(2k-1)x + \cos(2k+1)x$ ***M1***

$= \frac{\sin 2kx}{2\sin x} + \cos(2k+1)x$ ***A1***

$= \frac{\sin 2kx + 2\cos(2k+1)x\sin x}{2\sin x}$ ***M1***

$= \frac{\sin(2k+1)x\cos x - \cos(2k+1)x\sin x + 2\cos(2k+1)x\sin x}{2\sin x}$ ***M1***

$= \frac{\sin(2k+1)x\cos x + \cos(2k+1)x\sin x}{2\sin x}$ ***A1***

$= \frac{\sin(2k+2)x}{2\sin x}$ ***M1***

$= \frac{\sin 2(k+1)x}{2\sin x}$ ***A1***

so if true for n = k, then also true for n = k + 1

as true for n = 1 then true for all $n \in \mathbb{Z}^+$ ***R1***

Note: Final ***R1*** is independent of previous work.

[12 marks]

(c) $\frac{\sin 4x}{2\sin x} = \frac{1}{2}$ ***M1A1***

$\sin 4x = \sin x$

$4x = x \Rightarrow x = 0$ but this is impossible

$4x = \pi - x \Rightarrow x = \frac{\pi}{5}$ ***A1***

$4x = 2\pi + x \Rightarrow x = \frac{2\pi}{3}$ ***A1***

$4x = 3\pi - x \Rightarrow x = \frac{3\pi}{5}$ ***A1***

for not including any answers outside the domain ***R1***

Note: Award the first ***M1A1*** for correctly obtaining $8\cos^3 x - 4\cos x - 1 = 0$ or equivalent and subsequent marks as appropriate including the answers $\left(-\frac{1}{2}, \frac{1 \pm \sqrt{5}}{4}\right)$.

[6 marks]

Total [20 marks]

Examiners report

This question showed the weaknesses of many candidates in dealing with formal proofs and showing their reasoning in a logical manner. In part (a) just a few candidates clearly showed the result and part (b) showed that most candidates struggle with the formality of a proof by induction. The logic of many solutions was poor, though sometimes contained correct trigonometric work. Very few candidates were successful in answering part (c) using the unit circle. Most candidates attempted to manipulate the equation to obtain a cubic equation but made little progress. A few candidates guessed $\frac{2\pi}{3}$ as a solution but were not able to determine the other solutions.

9. *[10 marks]*

Markscheme

(a) (i) $1 \times 2 + 2 \times 3 + \ldots + n(n+1) = \frac{1}{3}n(n+1)(n+2)$ ***R1***

(ii) LHS = 40; RHS = 40 ***A1***

[2 marks]

(b) the sequence of values are:

5, 7, 11, 19, 35 … or an example ***A1***

35 is not prime, so Bill’s conjecture is false ***R1AG***

[2 marks]

(c) $\mathrm{P}(n) : 5 \times 7^n + 1$ is divisible by 6

$\mathrm{P}(1) : 36$ is divisible by $6 \Rightarrow \mathrm{P}(1)$ true ***A1***

assume $\mathrm{P}(k)$ is true $(5 \times 7^k + 1 = 6r)$ ***M1***

Note: Do **not** award ***M1*** for statement starting 'let *n* = *k*'.

Subsequent marks are independent of this ***M1***.

consider $5 \times 7^{k+1} + 1$ ***M1***

$= 7(6r - 1) + 1$ ***(A1)***

$= 6(7r - 1) \Rightarrow \mathrm{P}(k + 1)$ is true ***A1***

P(1) true and $\mathrm{P}(k)$ true $\Rightarrow \mathrm{P}(k + 1)$ true, so by MI $\mathrm{P}(n)$ is true for all $n \in \mathbb{Z}^+$ ***R1***

Note: Only award ***R1*** if there is consideration of P(1), $\mathrm{P}(k)$ and $\mathrm{P}(k + 1)$ in the final statement.

Only award ***R1*** if at least one of the two preceding ***A*** marks has been awarded.

[6 marks]

Total [10 marks]

Examiners report

Although there were a good number of wholly correct solutions to this question, it was clear that a number of students had not been prepared for questions on conjectures. The proof by induction was relatively well done, but candidates often showed a lack of rigour in the proof. It was fairly common to see students who did not appreciate the idea that $\mathrm{P}(k)$ is assumed not given and this was penalised. Also it appeared that a number of students had been taught to write down the final reasoning for a proof by induction, even if no attempt of a proof had taken place. In these cases, the final reasoning mark was not awarded.

10. *[7 marks]*

Markscheme

$\mathrm{P}(n) : f(n) = 5^{2n} - 24n - 1$ is divisible by 576 for $n \in \mathbb{Z}^+$

for $n = 1,\ f(1) = 5^2 - 24 - 1 = 0$

Zero is divisible by 576, (as every non-zero number divides zero), and so P(1) is true. ***R1***

Note: Award ***R0*** for P(1) = 0 shown and zero is divisible by 576 not specified.

Note: Ignore P(2) = 576 if P(1) = 0 is shown and zero is divisible by 576 is specified.

Assume $\mathrm{P}(k)$ is true for some $k\ (\Rightarrow f(k) = N \times 576)$. ***M1***

Note: Do not award ***M1*** for statements such as "let *n* = *k*".

consider $\mathrm{P}(k + 1) : f(k + 1) = 5^{2(k+1)} - 24(k + 1) - 1$ ***M1***

$= 25 \times 5^{2k} - 24k - 25$ ***A1***

EITHER

$= 25 \times (24k + 1 + N \times 576) - 24k - 25$ ***A1***

$= 576k + 25 \times 576N$ which is a multiple of 576 ***A1***

OR

$= 25 \times 5^{2k} - 600k - 25 + 600k - 24k$ ***A1***

$= 25(5^{2k} - 24k - 1) + 576k$ (or equivalent) which is a multiple of 576 ***A1***

THEN

P(1) is true and $\mathrm{P}(k)$ true $\Rightarrow \mathrm{P}(k+1)$ true, so $\mathrm{P}(n)$ is true for all $n \in \mathbb{Z}^+$ ***R1***

Note: Award ***R1*** only if at least four prior marks have been awarded.

[7 marks]

Examiners report

This proof by mathematical induction challenged most candidates. While most candidates were able to show that P(1) = 0, a significant number did not state that zero is divisible by 576. A few candidates started their proof by looking at P(2). It was pleasing to see that the inductive step was reasonably well done by most candidates. However many candidates committed simple algebraic errors. The most common error was to state that $5^{2(k+1)} = 5(5)^{2k}$. The concluding statement often omitted the required implication statement and also often omitted that P(1) was found to be true.

11. *[7 marks]*

Markscheme

proposition is true for *n* = 1 since $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{(1-x)^2}$ ***M1***

$= \frac{1!}{(1-x)^2}$ ***A1***

Note: Must see the 1! for the ***A1***.

assume true for *n* = *k*, $k \in \mathbb{Z}^+$, *i.e.* $\frac{\mathrm{d}^k y}{\mathrm{d}x^k} = \frac{k!}{(1-x)^{k+1}}$ ***M1***

consider $\frac{\mathrm{d}^{k+1}y}{\mathrm{d}x^{k+1}} = \frac{\mathrm{d}\left(\frac{\mathrm{d}^k y}{\mathrm{d}x^k}\right)}{\mathrm{d}x}$ ***(M1)***

$= (k+1)k!(1-x)^{-(k+1)-1}$ ***A1***

$= \frac{(k+1)!}{(1-x)^{k+2}}$ ***A1***

hence, P_{k+1} is true whenever P_k is true, and P_1 is true, and therefore the proposition is true for all positive integers ***R1***

Note: The final ***R1*** is only available if at least 4 of the previous marks have been awarded.

[7 marks]

Examiners report

Most candidates were awarded good marks for this question. A disappointing minority thought that the $(k+1)$th derivative was the (k)th derivative multiplied by the first derivative. Providing an acceptable final statement remains a perennial issue.

12. *[8 marks]*

Markscheme

if $n = 0$

$7^3 + 2 = 345$ which is divisible by 5, hence true for $n = 0$ ***A1***

Note: Award ***A0*** for using $n = 1$ but do not penalize further in question.

assume true for $n = k$ ***M1***

Note: Only award the ***M1*** if truth is assumed.

so $7^{8k+3} + 2 = 5p,\ p \in \bullet$ ***A1***

if $n = k + 1$

$7^{8(k+1)+3} + 2$ ***M1***

$= 7^8 7^{8k+3} + 2$ ***M1***

$= 7^8(5p - 2) + 2$ ***A1***

$= 7^8.5p - 2.7^8 + 2$

$= 7^8.5p - 11\,529\,600$

$= 5(7^8 p - 2\,305\,920)$ ***A1***

hence if true for $n = k$, then also true for $n = k + 1$. Since true for $n = 0$, then true for all $n \in \bullet$ ***R1***

Note: Only award the ***R1*** if the first two ***M1***s have been awarded.

[8 marks]

Examiners report

[N/A]

13a. *[7 marks]*

Markscheme

let $\mathrm{P}(n)$ be the proposition $z^n = r^n(\cos n\theta + \mathrm{i}\sin n\theta), n \in ¢^+$

let $n = 1 \Rightarrow$

$\mathrm{LHS} = r(\cos\theta + \mathrm{i}\sin\theta)$

$\mathrm{RHS} = r(\cos\theta + \mathrm{i}\sin\theta), \quad \therefore \mathrm{P}(1)$ is true ***R1***

assume true for $n = k \Rightarrow r^k(\cos\theta + \mathrm{i}\sin\theta)^k = r^k\left(\cos(k\theta) + \mathrm{i}\sin(k\theta)\right)$ ***M1***

Note: Only award the ***M1*** if truth is assumed.

now show $n = k$ true implies $n = k + 1$ also true

$r^{k+1}(\cos\theta + \mathrm{i}\sin\theta)^{k+1} = r^{k+1}(\cos\theta + \mathrm{i}\sin\theta)^k(\cos\theta + \mathrm{i}\sin\theta)$ ***M1***

$= r^{k+1}\left(\cos(k\theta) + \mathrm{i}\sin(k\theta)\right)(\cos\theta + \mathrm{i}\sin\theta)$

$= r^{k+1}\left(\cos(k\theta)\cos\theta - \sin(k\theta)\sin\theta + \mathrm{i}\left(\sin(k\theta)\cos\theta + \cos(k\theta)\sin\theta\right)\right)$ ***A1***

$= r^{k+1}\left(\cos(k\theta + \theta) + \mathrm{i}\sin(k\theta + \theta)\right)$ ***A1***

$= r^{k+1}\left(\cos(k+1)\theta + \mathrm{i}\sin(k+1)\theta\right) \Rightarrow n = k + 1$ is true ***A1***

$\mathrm{P}(k)$ true implies $\mathrm{P}(k+1)$ true and $\mathrm{P}(1)$ is true, therefore by mathematical induction statement is true for $n \geqslant 1$ ***R1***

Note: Only award the final ***R1*** if the first 4 marks have been awarded.

[7 marks]

Examiners report

[N/A]

13b. *[4 marks]*

Markscheme

(i) $u = 2\mathrm{cis}\left(\frac{\pi}{3}\right)$ ***A1***

$v = \sqrt{2}\mathrm{cis}\left(-\frac{\pi}{4}\right)$ ***A1***

Notes: Accept 3 sf answers only. Accept equivalent forms.

Accept $2e^{\frac{\pi}{3}i}$ and $\sqrt{2}e^{-\frac{\pi}{4}i}$.

(ii) $u^3 = 2^3\mathrm{cis}(\pi) = -8$

$v^4 = 4\text{cis}(-\pi) = -4$ ***(M1)***

$u^3 v^4 = 32$ ***A1***

Notes: Award ***(M1)*** for an attempt to find u^3 and v^4.

Accept equivalent forms.

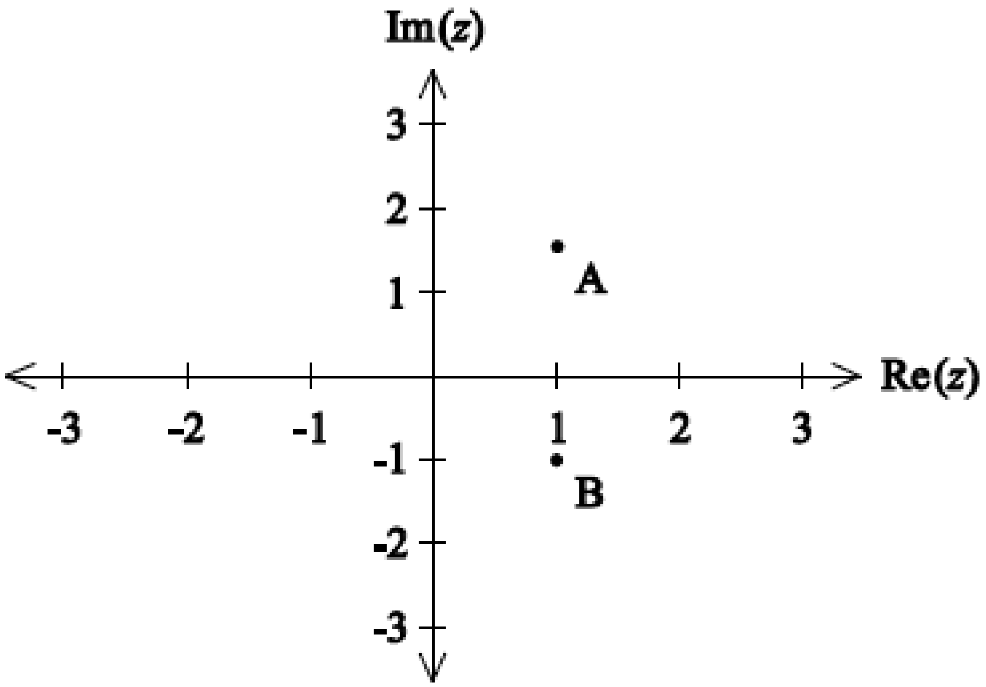
[4 marks]

Examiners report

[N/A]

13c. *[1 mark]*

Markscheme



A1

Note: Award **A1** if A or $1 + \sqrt{3}i$ and B or $1 - i$ are in their correct quadrants, are aligned vertically and it is clear that $|u| > |v|$.

[1 mark]

Examiners report

[N/A]

13d. *[3 marks]*

Markscheme

Area $= \frac{1}{2} \times 2 \times \sqrt{2} \times \sin\left(\frac{5\pi}{12}\right)$ ***M1A1***

$= 1.37 \left(= \frac{\sqrt{2}}{4}\left(\sqrt{6} + \sqrt{2}\right)\right)$ ***A1***

Notes: Award ***M1A0A0*** for using $\frac{7\pi}{12}$.

[3 marks]

Examiners report

[N/A]

13e. *[5 marks]*

Markscheme

$(z-1+\mathrm{i})(z-1-\mathrm{i})=z^2-2z+2$ ***M1A1***

Note: Award ***M1*** for recognition that a complex conjugate is also a root.

$\left(z-1-\sqrt{3}\mathrm{i}\right)\left(z-1+\sqrt{3}\mathrm{i}\right)=z^2-2z+4$ ***A1***

$\left(z^2-2z+2\right)\left(z^2-2z+4\right)=z^4-4z^3+10z^2-12z+8$ ***M1A1***

Note: Award ***M1*** for an attempt to expand two quadratics.

[5 marks]

Examiners report

[N/A]

14. *[7 marks]*

Markscheme

$n=1:\ 1^3+11=12$

$=3\times 4$ **or** a multiple of 3 ***A1***

assume the proposition is true for $n=k\ (ie\ k^3+11k=3\mathrm{m})$ ***M1***

Note: Do not award ***M1*** for statements with "Let $n=k$".

consider $n=k+1:(k+1)^3+11(k+1)$ ***M1***

$=k^3+3k^2+3k+1+11k+11$ ***A1***

$=k^3+11k+(3k^2+3k+12)$ ***M1***

$=3(m+k^2+k+4)$ ***A1***

Note: Accept $k^3+11k+3(k^2+k+4)$ or statement that $k^3+11k+(3k^2+3k+12)$ is a multiple of 3.

true for $n=1$, and $n=k$ true $\Rightarrow n=k+1$ true hence true for all $n \in \mathbb{Z}^+$ ***R1***

Note: Only award the final ***R1*** if at least 4 of the previous marks have been achieved.

[7 marks]

Examiners report

It was pleasing to see a great many clear and comprehensive answers for this relatively straightforward induction question. The inductive step only seemed to pose problems for the very weakest candidates. As in previous sessions, marks were mainly lost by candidates writing variations on ‘Let $n = k$’, rather than ‘Assume true for $n = k$’. The final reasoning step still needs attention, with variations on ‘$n = k + 1 \text{ true } \Rightarrow n = k \text{ true}$’ evident, suggesting that mathematical induction as a technique is not clearly understood.