Integration by Substitution and Parts 2008-2014 with MS

1a. *[5 marks]*

Let $f(x) = \sqrt{\frac{x}{1-x}},\ 0 < x < 1$.

Show that $f'(x) = \frac{1}{2}x^{-\frac{1}{2}}(1-x)^{-\frac{3}{2}}$ and deduce that *f* is an increasing function.

1b. *[6 marks]*

Show that the curve $y = f(x)$ has one point of inflexion, and find its coordinates.

1c. *[11 marks]*

Use the substitution $x = \sin^2\theta$ to show that $\int f(x)\mathrm{d}x = \arcsin\sqrt{x} - \sqrt{x - x^2} + c$.

2a. *[1 mark]*

The function *f* is defined on the domain $\left[0,\ \frac{3\pi}{2}\right]$ by $f(x) = e^{-x}\cos x$.

State the two zeros of *f*.

2b. *[1 mark]*

Sketch the graph of *f*.

2c. *[7 marks]*

The region bounded by the graph, the *x*-axis and the *y*-axis is denoted by *A* and the region bounded by the graph and the *x*-axis is denoted by *B* . Show that the ratio of the area of *A* to the area of *B* is

$$\frac{e^{\pi}\left(e^{\frac{\pi}{2}} + 1\right)}{e^{\pi} + 1}.$$

3. *[7 marks]*

By using the substitution $x = \sin t$, find $\int \frac{x^3}{\sqrt{1-x^2}}\mathrm{d}x$.

4. *[6 marks]*

By using an appropriate substitution find

$$\int \frac{\tan(\ln y)}{y}\mathrm{d}y,\ y > 0.$$

5. *[6 marks]*

Show that $\int_0^{\frac{\pi}{6}} x\sin 2x\mathrm{d}x = \frac{\sqrt{3}}{8} - \frac{\pi}{24}$.

6. *[5 marks]*

Calculate the exact value of $\int_1^{\mathrm{e}} x^2\ln x\mathrm{d}x$.

7a. *[9 marks]*

(i) Sketch the graphs of $y = \sin x$ and $y = \sin 2x$, on the same set of axes, for $0 \leqslant x \leqslant \frac{\pi}{2}$.

(ii) Find the x-coordinates of the points of intersection of the graphs in the domain $0 \leqslant x \leqslant \frac{\pi}{2}$.

(iii) Find the area enclosed by the graphs.

7b. *[8 marks]*

Find the value of $\int_0^1 \sqrt{\frac{x}{4-x}}\mathrm{d}x$ using the substitution $x = 4\sin^2\theta$.

7c. *[8 marks]*

The increasing function *f* satisfies $f(0) = 0$ and $f(a) = b$, where $a > 0$ and $b > 0$.

(i) By reference to a sketch, show that $\int_0^a f(x)\mathrm{d}x = ab - \int_0^b f^{-1}(x)\mathrm{d}x$.

(ii) **Hence** find the value of $\int_0^2 \arcsin\left(\frac{x}{4}\right)\mathrm{d}x$.

8a. *[8 marks]*

Prove by mathematical induction that, for $n \in \mathbb{Z}^+$,

$$1 + 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + 4\left(\frac{1}{2}\right)^3 + \ldots + n\left(\frac{1}{2}\right)^{n-1} = 4 - \frac{n+2}{2^{n-1}}.$$

8b. *[17 marks]*

(a) Using integration by parts, show that $\int \mathrm{e}^{2x}\sin x\mathrm{d}x = \frac{1}{5}\mathrm{e}^{2x}(2\sin x - \cos x) + C$.

(b) Solve the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \sqrt{1-y^2}\mathrm{e}^{2x}\sin x$, given that *y* = 0 when *x* = 0,

writing your answer in the form $y = f(x)$.

(c) (i) Sketch the graph of $y = f(x)$, found in part (b), for $0 \leqslant x \leqslant 1.5$.

Determine the coordinates of the point P, the first positive intercept on the *x*-axis, and mark it on your sketch.

(ii) The region bounded by the graph of $y = f(x)$ and the *x*-axis, between the origin and P, is rotated 360° about the *x*-axis to form a solid of revolution.

Calculate the volume of this solid.

9a. *[6 marks]*

The integral I_n is defined by $I_n = \int_{n\pi}^{(n+1)\pi} \mathrm{e}^{-x}|\sin x|\mathrm{d}x, \text{ for } n \in \mathbb{N}$.

Show that $I_0 = \frac{1}{2}(1 + \mathrm{e}^{-\pi})$.

9b. *[4 marks]*

By letting $y = x - n\pi$, show that $I_n = \mathrm{e}^{-n\pi}I_0$.

9c. *[5 marks]*

Hence determine the exact value of $\int_0^\infty \mathrm{e}^{-x}|\sin x|\mathrm{d}x$.

10a. *[6 marks]*

Calculate $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec^2 x}{\sqrt[3]{\tan x}}\mathrm{d}x$.

10b. *[3 marks]*

Find $\int \tan^3 x\mathrm{d}x$.

11a. *[7 marks]*

Find the value of the integral $\int_0^{\sqrt{2}} \sqrt{4-x^2}\mathrm{d}x$.

11b. *[5 marks]*

Find the value of the integral $\int_0^{0.5} \arcsin x\mathrm{d}x$.

11c. *[7 marks]*

Using the substitution $t = \tan\theta$, find the value of the integral

$$\int_0^{\frac{\pi}{4}} \frac{d\theta}{3\cos^2\theta + \sin^2\theta}.$$

12a. *[2 marks]*

Express $4x^2 - 4x + 5$ in the form $a(x-h)^2 + k$ where *a*, *h*, $k \in \mathbb{Q}$.

12b. *[3 marks]*

The graph of $y = x^2$ is transformed onto the graph of $y = 4x^2 - 4x + 5$. Describe a sequence of transformations that does this, making the order of transformations clear.

12c. *[2 marks]*

The function *f* is defined by $f(x) = \frac{1}{4x^2-4x+5}$.

Sketch the graph of $y = f(x)$.

12d. *[2 marks]*

Find the range of *f*.

12e. *[3 marks]*

By using a suitable substitution show that $\int f(x)dx = \frac{1}{4}\int \frac{1}{u^2+1}du$.

12f. *[7 marks]*

Prove that $\int_1^{3.5} \frac{1}{4x^2-4x+5}dx = \frac{\pi}{16}$.

13. *[7 marks]*

(a) Given that $\alpha > 1$, use the substitution $u = \frac{1}{x}$ to show that

$$\int_1^{\alpha} \frac{1}{1+x^2}dx = \int_{\frac{1}{\alpha}}^{1} \frac{1}{1+u^2}dx.$$

(b) **Hence** show that $\arctan\alpha + \arctan\frac{1}{\alpha} = \frac{\pi}{2}$.

14. *[6 marks]*

Find the value of $\int_0^1 t\ln(t+1)dt$.

15a. *[4 marks]*

Find $\int x\sec^2 x dx$.

15b. *[2 marks]*

Determine the value of *m* if $\int_0^m x\sec^2 x dx = 0.5$, where *m* > 0.

16. *[5 marks]*

(a) Integrate $\int \frac{\sin\theta}{1-\cos\theta}d\theta$.

(b) Given that $\int_{\frac{\pi}{2}}^{a} \frac{\sin\theta}{1-\cos\theta}d\theta = \frac{1}{2}$ and $\frac{\pi}{2} < a < \pi$, find the value of a.

17. *[8 marks]*

Using the substitution $x = 2\sin\theta$, show that

$$\int\sqrt{4-x^2}dx = Ax\sqrt{4-x^2} + B\arcsin\frac{x}{2} + \text{constant},$$

where

A

and

B

are constants whose values you are required to find.

18a. *[5 marks]*

Let $f(x) = x(x+2)^6$.

Solve the inequality $f(x) > x$.

18b. *[5 marks]*

Find $\int f(x)\mathrm{d}x$.

19. *[7 marks]*

Use the substitution $x = a\sec\theta$ to show that $\int_{a\sqrt{2}}^{2a} \frac{\mathrm{d}x}{x^3\sqrt{x^2-a^2}} = \frac{1}{24a^3}\left(3\sqrt{3}+\pi-6\right)$.

20a. *[2 marks]*

Particle *A* moves such that its velocity $v\mathrm{m\ s}^{-1}$, at time *t* seconds, is given by $v(t) = \frac{t}{12+t^4},\ t \geqslant 0$.

Sketch the graph of $y = v(t)$. Indicate clearly the local maximum and write down its coordinates.

20b. *[4 marks]*

Use the substitution $u = t^2$ to find $\int \frac{t}{12+t^4}\,\mathrm{d}t$.

20c. *[3 marks]*

Find the exact distance travelled by particle A between $t = 0$ and $t = 6$ seconds.

Give your answer in the form $k\arctan(b),\ k,\ b \in \mathbb{R}$.

20d. *[3 marks]*

Particle *B* moves such that its velocity $v\mathrm{m\ s}^{-1}$ is related to its displacement $s\mathrm{m}$, by the equation $v(s) = \arcsin\left(\sqrt{s}\right)$.

Find the acceleration of particle B when $s = 0.1\mathrm{m}$.

21. *[7 marks]*

By using the substitution $x = 2\tan u$, show that $\int \frac{\mathrm{d}x}{x^2\sqrt{x^2+4}} = \frac{-\sqrt{x^2+4}}{4x} + C$.

Integration by Substitution and Parts 2008-2014 MS

1a. *[5 marks]*

Markscheme

EITHER

derivative of $\frac{x}{1-x}$ is $\frac{(1-x)-x(-1)}{(1-x)^2}$ ***M1A1***

$f'(x) = \frac{1}{2}\left(\frac{x}{1-x}\right)^{-\frac{1}{2}} \frac{1}{(1-x)^2}$ ***M1A1***

$= \frac{1}{2}x^{-\frac{1}{2}}(1-x)^{-\frac{3}{2}}$ ***AG***

$f'(x) > 0$ (for all $0 < x < 1$) so the function is increasing ***R1***

OR

$f(x) = \frac{x^{\frac{1}{2}}}{(1-x)^{\frac{1}{2}}}$

$f'(x) = \frac{(1-x)^{\frac{1}{2}}\left(\frac{1}{2}x^{-\frac{1}{2}}\right) - \frac{1}{2}x^{\frac{1}{2}}(1-x)^{-\frac{1}{2}}(-1)}{1-x}$ ***M1A1***

$= \frac{1}{2}x^{-\frac{1}{2}}(1-x)^{-\frac{1}{2}} + \frac{1}{2}x^{\frac{1}{2}}(1-x)^{-\frac{3}{2}}$ ***A1***

$= \frac{1}{2}x^{-\frac{1}{2}}(1-x)^{-\frac{3}{2}}[1-x+x]$ ***M1***

$= \frac{1}{2}x^{-\frac{1}{2}}(1-x)^{-\frac{3}{2}}$ ***AG***

$f'(x) > 0$ (for all $0 < x < 1$) so the function is increasing ***R1***

[5 marks]

1b. *[6 marks]*

Markscheme

$f'(x) = \frac{1}{2}x^{-\frac{1}{2}}(1-x)^{-\frac{3}{2}}$

$\Rightarrow f''(x) = -\frac{1}{4}x^{-\frac{3}{2}}(1-x)^{-\frac{3}{2}} + \frac{3}{4}x^{-\frac{1}{2}}(1-x)^{-\frac{5}{2}}$ ***M1A1***

$= -\frac{1}{4}x^{-\frac{3}{2}}(1-x)^{-\frac{5}{2}}[1-4x]$

$f''(x) = 0 \Rightarrow x = \frac{1}{4}$ ***M1A1***

$f''(x)$ changes sign at $x = \frac{1}{4}$ hence there is a point of inflexion ***R1***

$x = \frac{1}{4} \Rightarrow y = \frac{1}{\sqrt{3}}$ ***A1***

the coordinates are $\left(\frac{1}{4}, \frac{1}{\sqrt{3}}\right)$

[6 marks]

1c. *[11 marks]*

Markscheme

$x = \sin^2\theta \Rightarrow \frac{dx}{d\theta} = 2\sin\theta\cos\theta$ ***M1A1***

$\int\sqrt{\frac{x}{1-x}}dx = \int\sqrt{\frac{\sin^2\theta}{1-\sin^2\theta}}2\sin\theta\cos\theta d\theta$ ***M1A1***

$= \int 2\sin^2\theta d\theta$ ***A1***

$= \int 1 - \cos 2\theta d\theta$ ***M1A1***

$= \theta - \frac{1}{2}\sin 2\theta + c$ ***A1***

$\theta = \arcsin\sqrt{x}$ ***A1***

$\frac{1}{2}\sin 2\theta = \sin\theta\cos\theta = \sqrt{x}\sqrt{1-x} = \sqrt{x-x^2}$ ***M1A1***

hence $\int\sqrt{\frac{x}{1-x}}dx = \arcsin\sqrt{x} - \sqrt{x-x^2} + c$ ***AG***

[11 marks]

Examiners report

Part (a) was generally well done, although few candidates made the final deduction asked for. Those that lost other marks in this part were generally due to mistakes in algebraic manipulation. In part

(b) whilst many students found the second derivative and set it equal to zero, few then confirmed that it was a point of inflexion. There were several good attempts for part (c), even though there were various points throughout the question that provided stopping points for other candidates.

2a. *[1 mark]*

Markscheme

$e^{-x} \cos x = 0$

$\Rightarrow x = \frac{\pi}{2}, \ \frac{3\pi}{2}$ ***A1***

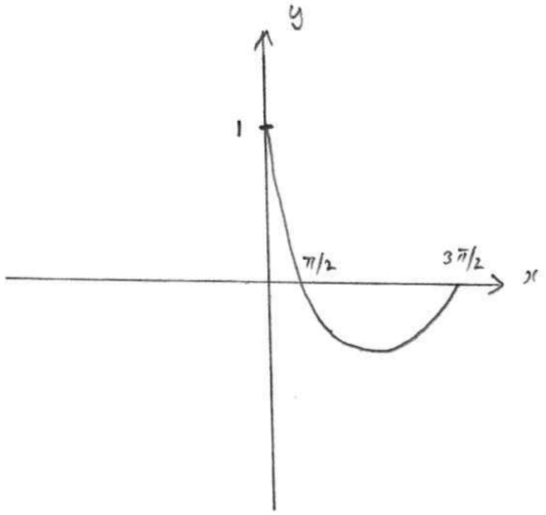
[1 mark]

Examiners report

Many candidates stated the two zeros of *f* correctly but the graph of *f* was often incorrectly drawn. In (c), many candidates failed to realise that integration by parts had to be used twice here and even those who did that often made algebraic errors, usually due to the frequent changes of sign.

2b. *[1 mark]*

Markscheme



A1

Note: Accept any form of concavity for $x \in \left[0, \frac{\pi}{2}\right]$.

Note: Do not penalize unmarked zeros if given in part (a).

Note: Zeros written on diagram can be used to allow the mark in part (a) to be awarded retrospectively.

[1 mark]

Examiners report

Many candidates stated the two zeros of *f* correctly but the graph of *f* was often incorrectly drawn. In (c), many candidates failed to realise that integration by parts had to be used twice here and even those who did that often made algebraic errors, usually due to the frequent changes of sign.

2c. *[7 marks]*

Markscheme

attempt at integration by parts ***M1***

EITHER

$I = \int \mathrm{e}^{-x} \cos x \mathrm{d}x = -\mathrm{e}^{-x} \cos x \mathrm{d}x - \int \mathrm{e}^{-x} \sin x \mathrm{d}x$ ***A1***

$\Rightarrow I = -\mathrm{e}^{-x} \cos x \mathrm{d}x - \left[-\mathrm{e}^{-x} \sin x + \int \mathrm{e}^{-x} \cos x \mathrm{d}x\right]$ ***A1***

$\Rightarrow I = \frac{\mathrm{e}^{-x}}{2}(\sin x - \cos x) + C$ ***A1***

Note: Do not penalize absence of *C*.

OR

$I = \int \mathrm{e}^{-x} \cos x \mathrm{d}x = \mathrm{e}^{-x} \sin x + \int \mathrm{e}^{-x} \sin x \mathrm{d}x$ ***A1***

$I = \mathrm{e}^{-x} \sin x - \mathrm{e}^{-x} \cos x - \int \mathrm{e}^{-x} \cos x \mathrm{d}x$ ***A1***

$\Rightarrow I = \frac{\mathrm{e}^{-x}}{2}(\sin x - \cos x) + C$ ***A1***

Note: Do not penalize absence of *C*.

THEN

$\int_0^{\frac{\pi}{2}} \mathrm{e}^{-x} \cos x \mathrm{d}x = \left[\frac{\mathrm{e}^{-x}}{2}(\sin x - \cos x)\right]_0^{\frac{\pi}{2}} = \frac{\mathrm{e}^{-\frac{\pi}{2}}}{2} + \frac{1}{2}$ ***A1***

$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \mathrm{e}^{-x} \cos x \mathrm{d}x = \left[\frac{\mathrm{e}^{-x}}{2}(\sin x - \cos x)\right]_{\frac{\pi}{2}}^{\frac{3\pi}{2}} = -\frac{\mathrm{e}^{-\frac{3\pi}{2}}}{2} - \frac{\mathrm{e}^{-\frac{\pi}{2}}}{2}$ ***A1***

ratio of A:B is $\frac{\frac{\mathrm{e}^{-\frac{\pi}{2}}}{2}+\frac{1}{2}}{\frac{\mathrm{e}^{-\frac{3\pi}{2}}}{2}+\frac{\mathrm{e}^{-\frac{\pi}{2}}}{2}}$

$= \frac{\mathrm{e}^{\frac{3\pi}{2}}\left(\mathrm{e}^{-\frac{\pi}{2}}+1\right)}{\mathrm{e}^{\frac{3\pi}{2}}\left(\mathrm{e}^{-\frac{3\pi}{2}}+\mathrm{e}^{-\frac{\pi}{2}}\right)}$ ***M1***

$= \frac{\mathrm{e}^{\pi}\left(\mathrm{e}^{\frac{\pi}{2}}+1\right)}{\mathrm{e}^{\pi}+1}$ ***AG***

[7 marks]

Examiners report

Many candidates stated the two zeros of *f* correctly but the graph of *f* was often incorrectly drawn. In (c), many candidates failed to realise that integration by parts had to be used twice here and even those who did that often made algebraic errors, usually due to the frequent changes of sign.

3. *[7 marks]*

Markscheme

$x = \sin t, \mathrm{d}\,x = \cos t\,\mathrm{d}t$

$\int \frac{x^3}{\sqrt{1-x^2}}\,\mathrm{d}x = \int \frac{\sin^3 t}{\sqrt{1-\sin^2 t}}\cos t\,\mathrm{d}t$ ***M1***

$= \int \sin^3 t\,\mathrm{d}t$ ***(A1)***

$= \int \sin^2 t \sin t\,\mathrm{d}t$

$= \int (1-\cos^2 t)\sin t\,\mathrm{d}t$ ***M1A1***

$= \int \sin t\,\mathrm{d}t - \int \cos^2 t \sin t\,\mathrm{d}t$

$= -\cos t + \frac{\cos^3 t}{3} + C$ ***A1A1***

$= -\sqrt{1-x^2} + \frac{1}{3}\left(\sqrt{1-x^2}\right)^3 + C$ ***A1***

$\left(= -\sqrt{1-x^2}\left(1-\frac{1}{3}(1-x^2)\right)+C\right)$

$\left(= -\frac{1}{3}\sqrt{1-x^2}(2+x^2)+C\right)$

[7 marks]

Examiners report

Just a few candidates got full marks in this question. Substitution was usually incorrectly done and lead to wrong results. A cosine term in the denominator was a popular error. Candidates often chose unhelpful trigonometric identities and attempted integration by parts. Results such as $\int \sin^3 t\,\mathrm{d}t = \frac{\sin^4 t}{4} + C$ were often seen along with other misconceptions concerning the manipulation/simplification of integrals were also noticed. Some candidates unsatisfactorily attempted to use $\arcsin x$. However, there were some good solutions involving an expression for the cube of $\sin t$ in terms of $\sin t$ and $\sin 3t$. Very few candidates re-expressed their final result in terms of *x*.

4. *[6 marks]*

Markscheme

Let $u = \ln y \Rightarrow \mathrm{d}u = \frac{1}{y}\,\mathrm{d}y$ ***A1(A1)***

$\int \frac{\tan(\ln y)}{y}\,\mathrm{d}y = \int \tan u \mathrm{d}u$ ***A1***

$= \int \frac{\sin u}{\cos u}\,\mathrm{d}u = -\ln|\cos u| + c$ ***A1***

EITHER

$\int \frac{\tan(\ln y)}{y}\,\mathrm{d}y = -\ln|\cos(\ln y)| + c$ ***A1A1***

OR

$\int \frac{\tan(\ln y)}{y}\,\mathrm{d}y = \ln|\sec(\ln y)| + c$ ***A1A1***

[6 marks]

Examiners report

Many candidates obtained the first three marks, but then attempted various methods unsuccessfully. Quite a few candidates attempted integration by parts rather than substitution. The candidates who successfully integrated the expression often failed to put the absolute value sign in the final answer.

5. *[6 marks]*

Markscheme

Using integration by parts ***(M1)***

$u = x,\ \frac{\mathrm{d}u}{\mathrm{d}x} = 1,\ \frac{\mathrm{d}v}{\mathrm{d}x} = \sin 2x \text{ and } v = -\frac{1}{2}\cos 2x$ ***(A1)***

$\left[x\left(-\frac{1}{2}\cos 2x\right)\right]_0^{\frac{\pi}{6}} - \int_0^{\frac{\pi}{6}} \left(-\frac{1}{2}\cos 2x\right)\mathrm{d}x$ ***A1***

$= \left[x\left(-\frac{1}{2}\cos 2x\right)\right]_0^{\frac{\pi}{6}} + \left[\frac{1}{4}\sin 2x\right]_0^{\frac{\pi}{6}}$ ***A1***

Note: Award the ***A1A1*** above if the limits are not included.

$\left[x\left(-\frac{1}{2}\cos 2x\right)\right]_0^{\frac{\pi}{6}} = -\frac{\pi}{24}$ ***A1***

$\left[\frac{1}{4}\sin 2x\right]_0^{\frac{\pi}{6}} = \frac{\sqrt{3}}{8}$ ***A1***

$\int_0^{\frac{\pi}{6}} x\sin 2x\mathrm{d}x = \frac{\sqrt{3}}{8} - \frac{\pi}{24}$ ***AG N0***

Note: Allow ***FT*** on the last two ***A1*** marks if the expressions are the negative of the correct ones.

[6 marks]

Examiners report

This question was reasonably well done, with few candidates making the inappropriate choice of *u* and $\frac{\mathrm{d}v}{\mathrm{d}x}$. The main source of a loss of marks was in finding *v* by integration. A few candidates used the double angle formula for sine, with poor results.

6. *[5 marks]*

Markscheme

Recognition of integration by parts ***M1***

$\int x^2 \ln x\mathrm{d}x = \left[\frac{x^3}{3}\ln x\right] - \int \frac{x^3}{3} \times \frac{1}{x}\mathrm{d}x$ ***A1A1***

$= \left[\frac{x^3}{3}\ln x\right] - \int \frac{x^2}{3}\mathrm{d}x$

$= \left[\frac{x^3}{3}\ln x - \frac{x^3}{9}\right]$ ***A1***

$\Rightarrow \int_1^{\mathrm{e}} x^2 \ln x\mathrm{d}x = \left(\frac{\mathrm{e}^3}{3} - \frac{\mathrm{e}^3}{9}\right) - \left(0 - \frac{1}{9}\right) \quad \left(= \frac{2\mathrm{e}^3+1}{9}\right)$ ***A1***

[5 marks]

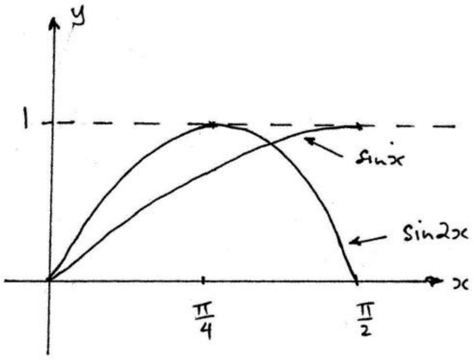
Examiners report

Most candidates recognised that a method of integration by parts was appropriate for this question. However, although a good number of correct answers were seen, a number of candidates made algebraic errors in the process. A number of students were also unable to correctly substitute the limits.

7a. *[9 marks]*

Markscheme

(i)



A2

Note: Award ***A1*** for correct $\sin x$, ***A1*** for correct $\sin 2x$.

Note: Award ***A1A0*** for two correct shapes with $\frac{\pi}{2}$ and/or 1 missing.

Note: Condone graph outside the domain.

(ii) $\sin 2x = \sin x$, $0 \leqslant x \leqslant \frac{\pi}{2}$

$2\sin x\cos x - \sin x = 0$ ***M1***

$\sin x(2\cos x - 1) = 0$

$x = 0, \frac{\pi}{3}$ ***A1A1 N1N1***

(iii) area $= \int_0^{\frac{\pi}{3}} (\sin 2x - \sin x)\mathrm{d}x$ ***M1***

Note: Award ***M1*** for an integral that contains limits, not necessarily correct, with $\sin x$ and $\sin 2x$ subtracted in either order.

$= \left[-\frac{1}{2}\cos 2x + \cos x\right]_0^{\frac{\pi}{3}}$ ***A1***

$= \left(-\frac{1}{2}\cos\frac{2\pi}{3} + \cos\frac{\pi}{3}\right) - \left(-\frac{1}{2}\cos 0 + \cos 0\right)$ ***(M1)***

$= \frac{3}{4} - \frac{1}{2}$

$= \frac{1}{4}$ ***A1***

[9 marks]

Examiners report

A significant number of candidates did not seem to have the time required to attempt this question satisfactorily.

Part (a) was done quite well by most but a number found sketching the functions difficult, the most common error being poor labelling of the axes.

Part (ii) was done well by most the most common error being to divide the equation by $\sin x$ and so omit the *x* = 0 value. Many recognised the value from the graph and corrected this in their final solution.

The final part was done well by many candidates.

Many candidates found (b) challenging. Few were able to substitute the *dx* expression correctly and many did not even seem to recognise the need for this term. Those that did tended to be able to find the integral correctly. Most saw the need for the double angle expression although many did not change the limits successfully.

Few candidates attempted part c). Those who did get this far managed the sketch well and were able to explain the relationship required. Among those who gave a response to this many were able to get the result although a number made errors in giving the inverse function. On the whole those who got this far did it well.

7b. *[8 marks]*

Markscheme

$\int_0^1 \sqrt{\frac{x}{4-x}}\mathrm{d}x = \int_0^{\frac{\pi}{6}} \sqrt{\frac{4\sin^2\theta}{4-4\sin^2\theta}} \times 8\sin\theta\cos\theta\mathrm{d}\theta$ ***M1A1A1***

Note: Award ***M1*** for substitution and reasonable attempt at finding expression for d*x* in terms of $\mathrm{d}\theta$, first ***A1*** for correct limits, second ***A1*** for correct substitution for d*x* .

$\int_0^{\frac{\pi}{6}} 8\sin^2\theta\mathrm{d}\theta$ ***A1***

$\int_0^{\frac{\pi}{6}} 4 - 4\cos 2\theta\mathrm{d}\theta$ ***M1***

$= [4\theta - 2\sin 2\theta]_0^{\frac{\pi}{6}}$ ***A1***

$= \left(\frac{2\pi}{3} - 2\sin\frac{\pi}{3}\right) - 0$ ***(M1)***

$= \frac{2\pi}{3} - \sqrt{3}$ ***A1***

[8 marks]

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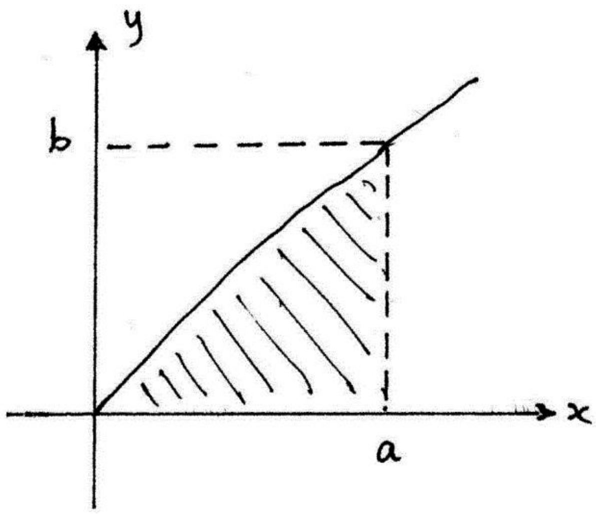
7c. *[8 marks]*

Markscheme

(i)



M1

from the diagram above

the shaded area $= \int_0^a f(x)\mathrm{d}x = ab - \int_0^b f^{-1}(y)\mathrm{d}y$ ***R1***

$= ab - \int_0^b f^{-1}(x)\mathrm{d}x$ ***AG***

(ii) $f(x) = \arcsin\frac{x}{4} \Rightarrow f^{-1}(x) = 4\sin x$ ***A1***

$\int_0^2 \arcsin\left(\frac{x}{4}\right)\mathrm{d}x = \frac{\pi}{3} - \int_0^{\frac{\pi}{6}} 4\sin x\mathrm{d}x$ ***M1A1A1***

Note: Award ***A1*** for the limit $\frac{\pi}{6}$ seen anywhere, ***A1*** for all else correct.

$= \frac{\pi}{3} - [-4\cos x]_0^{\frac{\pi}{6}}$ ***A1***

$= \frac{\pi}{3} - 4 + 2\sqrt{3}$ ***A1***

Note: Award no marks for methods using integration by parts.

[8 marks]

Examiners report

A significant number of candidates did not seem to have the time required to attempt this question satisfactorily.
Part (a) was done quite well by most but a number found sketching the functions difficult, the most common error being poor labelling of the axes.
Part (ii) was done well by most the most common error being to divide the equation by $\sin x$ and so omit the *x* = 0 value. Many recognised the value from the graph and corrected this in their final solution.
The final part was done well by many candidates.
Many candidates found (b) challenging. Few were able to substitute the *dx* expression correctly and many did not even seem to recognise the need for this term. Those that did tended to be able to find the integral correctly. Most saw the need for the double angle expression although many did not change the limits successfully.
Few candidates attempted part c). Those who did get this far managed the sketch well and were able to explain the relationship required. Among those who gave a response to this many were able to get the result although a number made errors in giving the inverse function. On the whole those who got this far did it well.

8a. *[8 marks]*

Markscheme

prove that $1+2\left(\frac{1}{2}\right)+3\left(\frac{1}{2}\right)^2+4\left(\frac{1}{2}\right)^3+\ldots+n\left(\frac{1}{2}\right)^{n-1}=4-\frac{n+2}{2^{n-1}}$
for *n* = 1
$\text{LHS}=1,\ \text{RHS}=4-\frac{1+2}{2^0}=4-3=1$
so true for *n* = 1 ***R1***
assume true for *n* = *k* ***M1***
so $1+2\left(\frac{1}{2}\right)+3\left(\frac{1}{2}\right)^2+4\left(\frac{1}{2}\right)^3+\ldots+k\left(\frac{1}{2}\right)^{k-1}=4-\frac{k+2}{2^{k-1}}$
now for *n* = *k* +1
LHS: $1+2\left(\frac{1}{2}\right)+3\left(\frac{1}{2}\right)^2+4\left(\frac{1}{2}\right)^3+\ldots+k\left(\frac{1}{2}\right)^{k-1}+(k+1)\left(\frac{1}{2}\right)^k$ ***A1***
$=4-\frac{k+2}{2^{k-1}}+(k+1)\left(\frac{1}{2}\right)^k$ ***M1A1***
$=4-\frac{2(k+2)}{2^k}+\frac{k+1}{2^k}$ (or equivalent) ***A1***
$=4-\frac{(k+1)+2}{2^{(k+1)-1}}$ (accept $4-\frac{k+3}{2^k}$) ***A1***
Therefore if it is true for *n* = *k* it is true for *n* = *k* + 1. It has been shown to be true for *n* = 1 so it is true for all $n\ (\in\mathbb{Z}^+)$. ***R1***
Note: To obtain the final ***R*** mark, a reasonable attempt at induction must have been made.
[8 marks]

Examiners report

Part A: Given that this question is at the easier end of the 'proof by induction' spectrum, it was disappointing that so many candidates failed to score full marks. The *n* = 1 case was generally well done. The whole point of the method is that it involves logic, so 'let n = k' or 'put n = k', instead of 'assume ... to be true for n = k', gains no marks. The algebraic steps need to be more convincing than some candidates were able to show. It is astonishing that the R1 mark for the final statement was so often not awarded.

8b. *[17 marks]*

Markscheme

(a)
METHOD 1
$\int e^{2x}\sin x\,dx=-\cos x e^{2x}+\int 2e^{2x}\cos x\,dx$ ***M1A1A1***
$=-\cos x e^{2x}+2e^{2x}\sin x-\int 4e^{2x}\sin x\,dx$ ***A1A1***
$5\int e^{2x}\sin x\,dx=-\cos x e^{2x}+2e^{2x}\sin x$ ***M1***
$\int e^{2x}\sin x\,dx=\frac{1}{5}e^{2x}(2\sin x-\cos x)+C$ ***AG***
METHOD 2
$\int \sin x e^{2x}\,dx=\frac{\sin x e^{2x}}{2}-\int\cos x\frac{e^{2x}}{2}\,dx$ ***M1A1A1***

$= \frac{\sin x e^{2x}}{2} - \cos x \frac{e^{2x}}{4} - \int \sin x \frac{e^{2x}}{4} dx$ ***A1A1***

$\frac{5}{4} \int e^{2x} \sin x dx = \frac{e^{2x} \sin x}{2} - \frac{\cos x e^{2x}}{4}$ ***M1***

$\int e^{2x} \sin x dx = \frac{1}{5} e^{2x} (2 \sin x - \cos x) + C$ ***AG***

[6 marks]

(b)

$\int \frac{dy}{\sqrt{1-y^2}} = \int e^{2x} \sin x dx$ ***M1A1***

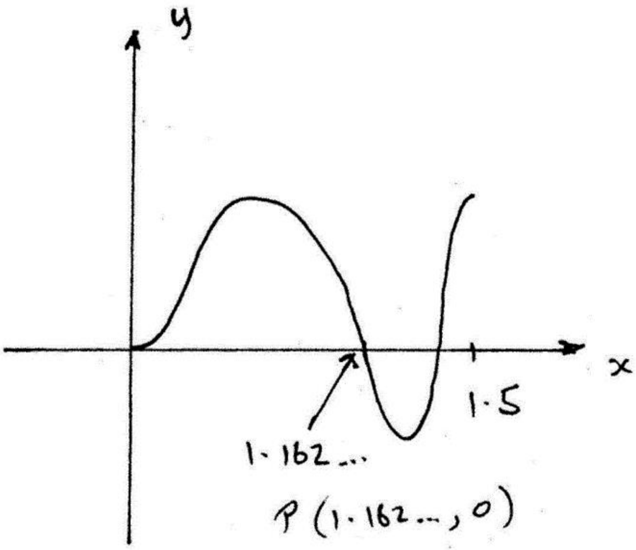
$\arcsin y = \frac{1}{5} e^{2x} (2 \sin x - \cos x)(+C)$ ***A1***

when $x = 0,\ y = 0 \Rightarrow C = \frac{1}{5}$ ***M1***

$y = \sin \left(\frac{1}{5} e^{2x} (2 \sin x - \cos x) + \frac{1}{5} \right)$ ***A1***

[5 marks]

(c)



(i) ***A1***

P is (1.16, 0) ***A1***

Note: Award ***A1*** for 1.16 seen anywhere, ***A1*** for complete sketch.

Note: Allow FT on their answer from (b)

(ii) $V = \int_0^{1.162...} \pi y^2 dx$ ***M1A1***

$= 1.05$ ***A2***

Note: Allow FT on their answers from (b) and (c)(i).

[6 marks]

Examiners report

Part B: Part (a) was often well done, although some faltered after the first integration. Part (b) was also generally well done, although there were some errors with the constant of integration. In (c) the graph was often attempted, but errors in (b) usually led to manifestly incorrect plots. Many attempted the volume of integration and some obtained the correct value.

9a. *[6 marks]*

Markscheme

$I_0 = \int_0^{\pi} e^{-x} \sin x dx$ ***M1***

Note: Award ***M1*** for $I_0 = \int_0^{\pi} e^{-x} |\sin x| dx$

Attempt at integration by parts, even if inappropriate modulus signs are present. ***M1***

$= -[e^{-x} \cos x]_0^{\pi} - \int_0^{\pi} e^{-x} \cos x dx$ **or** $= -[e^{-x} \sin x]_0^{\pi} - \int_0^{\pi} e^{-x} \cos x dx$ ***A1***

$= -[e^{-x} \cos x]_0^{\pi} - [e^{-x} \sin x]_0^{\pi} - \int_0^{\pi} e^{-x} \sin x dx$ **or**

$= -[e^{-x} \sin x + e^{-x} \cos x]_0^{\pi} - \int_0^{\pi} e^{-x} \sin x dx$ ***A1***

$= -[e^{-x} \cos x]_0^{\pi} - [e^{-x} \sin x]_0^{\pi} - I_0$ **or** $-[e^{-x} \sin x + e^{-x} \cos x]_0^{\pi} - I_0$ ***M1***

Note: Do not penalise absence of limits at this stage

$I_0 = e^{-\pi} + 1 - I_0$ ***A1***

$I_0 = \frac{1}{2}\left(1 + \mathrm{e}^{-\pi}\right)$ ***AG***

Note: If modulus signs are used around cos *x* , award no accuracy marks but do not penalise modulus signs around sin *x* .

[6 marks]

Examiners report

Part (a) is essentially core work requiring repeated integration by parts and many candidates realised that. However, some candidates left the modulus signs in I_0 which invalidated their work. In parts (b) and (c) it was clear that very few candidates had a complete understanding of the significance of the modulus sign and what conditions were necessary for it to be dropped. Overall, attempts at (b) and (c) were disappointing with few correct solutions seen.

9b. *[4 marks]*

Markscheme

$I_n = \int_{n\pi}^{(n+1)\pi} \mathrm{e}^{-x}|\sin x|\mathrm{d}x$

Attempt to use the substitution $y = x - n\pi$ ***M1***

(putting $y = x - n\pi$, $\mathrm{d}y = \mathrm{d}x$ and $[n\pi,\ (n+1)\pi] \to [0,\ \pi]$)

so $I_n = \int_0^{\pi} \mathrm{e}^{-(y+n\pi)}|\sin(y + n\pi)|\mathrm{d}y$ ***A1***

$= \mathrm{e}^{-n\pi} \int_0^{\pi} \mathrm{e}^{-y}|\sin(y + n\pi)|\mathrm{d}y$ ***A1***

$= \mathrm{e}^{-n\pi} \int_0^{\pi} \mathrm{e}^{-y} \sin y\mathrm{d}y$ ***A1***

$= \mathrm{e}^{-n\pi} I_0$ ***AG***

[4 marks]

Examiners report

Part (a) is essentially core work requiring repeated integration by parts and many candidates realised that. However, some candidates left the modulus signs in I_0 which invalidated their work. In parts (b) and (c) it was clear that very few candidates had a complete understanding of the significance of the modulus sign and what conditions were necessary for it to be dropped. Overall, attempts at (b) and (c) were disappointing with few correct solutions seen.

9c. *[5 marks]*

Markscheme

$\int_0^{\infty} \mathrm{e}^{-x}|\sin x|\mathrm{d}x = \sum_{n=0}^{\infty} I_n$ ***M1***

$= \sum_{n=0}^{\infty} \mathrm{e}^{-n\pi} I_0$ ***(A1)***

the $\sum$ term is an infinite geometric series with common ratio $\mathrm{e}^{-\pi}$ ***(M1)***

therefore

$\int_0^{\infty} \mathrm{e}^{-x}|\sin x|\mathrm{d}x = \frac{I_0}{1-\mathrm{e}^{-\pi}}$ ***(A1)***

$= \frac{1+\mathrm{e}^{-\pi}}{2(1-\mathrm{e}^{-\pi})} \quad \left(= \frac{\mathrm{e}^{\pi}+1}{2(\mathrm{e}^{\pi}-1)}\right)$ ***A1***

[5 marks]

Examiners report

Part (a) is essentially core work requiring repeated integration by parts and many candidates realised that. However, some candidates left the modulus signs in I_0 which invalidated their work. In parts (b) and (c) it was clear that very few candidates had a complete understanding of the significance of the modulus sign and what conditions were necessary for it to be dropped. Overall, attempts at (b) and (c) were disappointing with few correct solutions seen.

10a. *[6 marks]*

Markscheme

EITHER

let $u = \tan x; \mathrm{d}\,u = \sec^2 x\mathrm{d}x$ ***(M1)***

consideration of change of limits ***(M1)***

$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec^2 x}{\sqrt[3]{\tan x}}\,\mathrm{d}x = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{u^{\frac{1}{3}}}\,\mathrm{d}u$ ***(A1)***

Note: Do not penalize lack of limits.

$= \left[\frac{3u^{\frac{2}{3}}}{2}\right]_1^{\sqrt{3}}$ ***A1***

$= \frac{3\times\sqrt{3}^{\frac{2}{3}}}{2} - \frac{3}{2} = \left(\frac{3\sqrt[3]{3}-3}{2}\right)$ ***A1A1 N0***

OR

$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec^2 x}{\sqrt[3]{\tan x}}\,\mathrm{d}x = \left[\frac{3(\tan x)^{\frac{2}{3}}}{2}\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$ ***M2A2***

$= \frac{3\times\sqrt{3}^{\frac{2}{3}}}{2} - \frac{3}{2} = \left(\frac{3\sqrt[3]{3}-3}{2}\right)$ ***A1A1 N0***

[6 marks]

Examiners report

Quite a variety of methods were successfully employed to solve part (a).

10b. *[3 marks]*

Markscheme

$\int \tan^3 x\mathrm{d}x = \int \tan x(\sec^2 x - 1)\mathrm{d}x$ ***M1***

$= \int(\tan x \times \sec^2 x - \tan x)\mathrm{d}x$

$= \frac{1}{2}\tan^2 x - \ln|\sec x| + C$ ***A1A1***

Note: Do not penalize the absence of absolute value or *C*.

[3 marks]

Examiners report

Many candidates did not attempt part (b).

11a. *[7 marks]*

Markscheme

let $x = 2\sin\theta$ ***M1***

$\mathrm{d}x = 2\cos\theta\mathrm{d}\theta$ ***A1***

$I = \int_0^{\frac{\pi}{4}} 2\cos\theta \times 2\cos\theta\mathrm{d}\theta \quad \left(= 4\int_0^{\frac{\pi}{4}} \cos^2\theta\mathrm{d}\theta\right)$ ***A1A1***

Note: Award ***A1*** for limits and ***A1*** for expression.

$= 2\int_0^{\frac{\pi}{4}} (1+\cos 2\theta)\mathrm{d}\theta$ ***A1***

$= 2\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{4}}$ ***A1***

$= 1 + \frac{\pi}{2}$ ***A1***

[7 marks]

Examiners report

[N/A]

11b. *[5 marks]*

Markscheme

$I = [x\arcsin x]_0^{0.5} - \int_0^{0.5} x \times \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x$ ***M1A1A1***

$= [x\arcsin x]_0^{0.5} + \left[\sqrt{1-x^2}\right]_0^{0.5}$ ***A1***

$= \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1$ ***A1***

[5 marks]

Examiners report

[N/A]

11c. *[7 marks]*

Markscheme

$\mathrm{d}t = \sec^2\theta\mathrm{d}\theta, \quad \left[0, \frac{\pi}{4}\right] \to [0,\ 1]$ ***A1(A1)***

$I = \int_0^1 \frac{\frac{dt}{(1+t^2)}}{\frac{3}{(1+t^2)} + \frac{t^2}{(1+t^2)}}$ ***M1(A1)***

$= \int_0^1 \frac{dt}{3+t^2}$ ***A1***

$= \frac{1}{\sqrt{3}} \left[\arctan\left(\frac{x}{\sqrt{3}}\right)\right]_0^1$ ***A1***

$= \frac{\pi}{6\sqrt{3}}$ ***A1***

[7 marks]

Examiners report

[N/A]

12a. *[2 marks]*

Markscheme

$4(x-0.5)^2+4$ ***A1A1***

Note: ***A1*** for two correct parameters, ***A2*** for all three correct.

[2 marks]

Examiners report

This question covered many syllabus areas, completing the square, transformations of graphs, range, integration by substitution and compound angle formulae. There were many good solutions to parts (a) – (e).

12b. *[3 marks]*

Markscheme

translation $\begin{pmatrix} 0.5 \\ 0 \end{pmatrix}$ (allow "0.5 to the right") ***A1***

stretch parallel to *y*-axis, scale factor 4 (allow vertical stretch or similar) ***A1***

translation $\begin{pmatrix} 0 \\ 4 \end{pmatrix}$ (allow "4 up") ***A1***

Note: All transformations must state magnitude and direction.

Note: First two transformations can be in either order.

It could be a stretch followed by a single translation of $\begin{pmatrix} 0.5 \\ 4 \end{pmatrix}$. If the vertical translation is before the stretch it is $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

[3 marks]

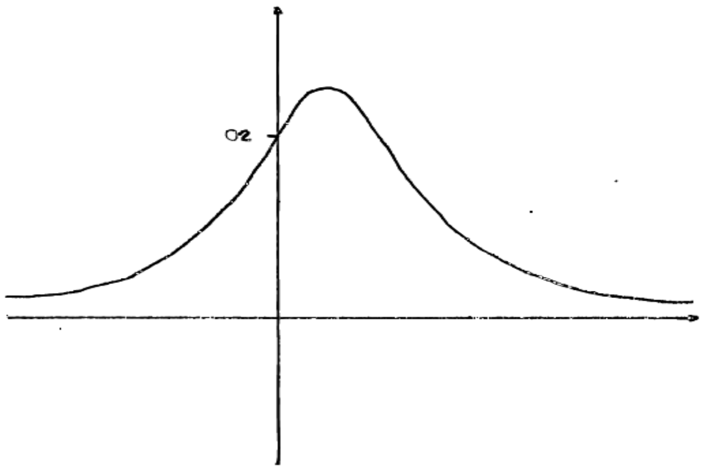
Examiners report

This question covered many syllabus areas, completing the square, transformations of graphs, range, integration by substitution and compound angle formulae. There were many good solutions to parts (a) – (e) but the following points caused some difficulties.

(b) Exam technique would have helped those candidates who could not get part (a) correct as any solution of the form given in the question could have led to full marks in part (b). Several candidates obtained expressions which were not of this form in (a) and so were unable to receive any marks in (b) Many missed the fact that if a vertical translation is performed before the vertical stretch it has a different magnitude to if it is done afterwards. Though on this occasion the markscheme was fairly flexible in the words it allowed to be used by candidates to describe the transformations it would be less risky to use the correct expressions.

12c. *[2 marks]*

Markscheme



general shape (including asymptote and single maximum in first quadrant), ***A1***

intercept $\left(0, \frac{1}{5}\right)$ or maximum $\left(\frac{1}{2}, \frac{1}{4}\right)$ shown ***A1***

[2 marks]

Examiners report

This question covered many syllabus areas, completing the square, transformations of graphs, range, integration by substitution and compound angle formulae. There were many good solutions to parts (a) – (e) but the following points caused some difficulties.

(c) Generally the sketches were poor. The general rule for all sketch questions should be that any asymptotes or intercepts should be clearly labelled. Sketches do not need to be done on graph paper, but a ruler should be used, particularly when asymptotes are involved.

12d. *[2 marks]*

Markscheme

$0 < f(x) \leqslant \frac{1}{4}$ ***A1A1***

Note: ***A1*** for $\leqslant \frac{1}{4}$, ***A1*** for $0 <$.

[2 marks]

Examiners report

This question covered many syllabus areas, completing the square, transformations of graphs, range, integration by substitution and compound angle formulae. There were many good solutions to parts (a) – (e).

12e. *[3 marks]*

Markscheme

let $u = x - \frac{1}{2}$ ***A1***

$\frac{\mathrm{d}u}{\mathrm{d}x} = 1 \quad (\text{or } \mathrm{d}u = \mathrm{d}x)$ ***A1***

$\int \frac{1}{4x^2-4x+5}\,\mathrm{d}x = \int \frac{1}{4\left(x-\frac{1}{2}\right)^2+4}\,\mathrm{d}x$ ***A1***

$\int \frac{1}{4u^2+4}\,\mathrm{d}u = \frac{1}{4}\int \frac{1}{u^2+1}\,\mathrm{d}u$ ***AG***

Note: If following through an incorrect answer to part (a), do not award final ***A1*** mark.

[3 marks]

Examiners report

This question covered many syllabus areas, completing the square, transformations of graphs, range, integration by substitution and compound angle formulae. There were many good solutions to parts (a) – (e) but the following points caused some difficulties.

(e) and (f) were well done up to the final part of (f), in which candidates did not realise they needed to use the compound angle formula.

12f. *[7 marks]*

Markscheme

$\int_1^{3.5} \frac{1}{4x^2-4x+5}\,\mathrm{d}x = \frac{1}{4}\int_{0.5}^{3} \frac{1}{u^2+1}\,\mathrm{d}u$ ***A1***

Note: ***A1*** for correct change of limits. Award also if they do not change limits but go back to *x* values when substituting the limit (even if there is an error in the integral).

$\frac{1}{4}\left[\arctan(u)\right]_{0.5}^{3}$ ***(M1)***
$\frac{1}{4}\left(\arctan(3)-\arctan\left(\frac{1}{2}\right)\right)$ ***A1***
let the integral = I
$\tan 4I=\tan\left(\arctan(3)-\arctan\left(\frac{1}{2}\right)\right)$ ***M1***
$\frac{3-0.5}{1+3\times 0.5}=\frac{2.5}{2.5}=1$ ***(M1)A1***
$4I=\frac{\pi}{4}\Rightarrow I=\frac{\pi}{16}$ ***A1AG***
[7 marks]

Examiners report

This question covered many syllabus areas, completing the square, transformations of graphs, range, integration by substitution and compound angle formulae. There were many good solutions to parts (a) – (e) but the following points caused some difficulties.
(e) and (f) were well done up to the final part of (f), in which candidates did not realise they needed to use the compound angle formula.

13. *[7 marks]*

Markscheme

(a) $u=\frac{1}{x}\Rightarrow \mathrm{d}u=-\frac{1}{x^2}\,\mathrm{d}x$ ***M1***
$\Rightarrow \mathrm{d}x=-\frac{\mathrm{d}u}{u^2}$ ***A1***
$\int_1^{\alpha}\frac{1}{1+x^2}\,\mathrm{d}x=-\int_1^{\frac{1}{\alpha}}\frac{1}{1+\left(\frac{1}{u}\right)^2}\frac{\mathrm{d}u}{u^2}$ ***A1M1A1***
Note: Award ***A1*** for correct integrand and ***M1A1*** for correct limits.
$=\int_{\frac{1}{\alpha}}^{1}\frac{1}{1+u^2}\,\mathrm{d}u$ (upon interchanging the two limits) ***AG***
(b) $\arctan x_1^{\alpha}=\arctan u_{\frac{1}{\alpha}}^{1}$ ***A1***
$\arctan\alpha-\frac{\pi}{4}=\frac{\pi}{4}-\arctan\frac{1}{\alpha}$ ***A1***
$\arctan\alpha+\arctan\frac{1}{\alpha}=\frac{\pi}{2}$ ***AG***
[7 marks]

Examiners report

This question was successfully answered by few candidates. Both parts of the question prescribed the approach which was required – “use the substitution” and “hence”. Many candidates ignored these. The majority of the candidates failed to use substitution properly to change the integration variables and in many cases the limits were fudged. The logic of part (b) was missing in many cases.

14. *[6 marks]*

Markscheme

EITHER
attempt at integration by substitution ***(M1)***
using $u=t+1,\ \mathrm{d}u=\mathrm{d}t$, the integral becomes
$\int_1^2(u-1)\ln u\,\mathrm{d}u$ ***A1***
then using integration by parts ***M1***
$\int_1^2(u-1)\ln u\,\mathrm{d}u=\left[\left(\frac{u^2}{2}-u\right)\ln u\right]_1^2-\int_1^2\left(\frac{u^2}{2}-u\right)\times\frac{1}{u}\,\mathrm{d}u$ ***A1***
$=-\left[\frac{u^2}{4}-u\right]_1^2$ ***(A1)***
$=\frac{1}{4}$ (accept 0.25) ***A1***
OR
attempt to integrate by parts ***(M1)***
correct choice of variables to integrate and differentiate ***M1***
$\int_0^1 t\ln(t+1)\mathrm{d}t=\left[\frac{t^2}{2}\ln(t+1)\right]_0^1-\int_0^1\frac{t^2}{2}\times\frac{1}{t+1}\,\mathrm{d}t$ ***A1***
$=\left[\frac{t^2}{2}\ln(t+1)\right]_0^1-\frac{1}{2}\int_0^1 t-1+\frac{1}{t+1}\,\mathrm{d}t$ ***A1***

$= \left[\frac{t^2}{2}\ln(t+1)\right]_0^1 - \frac{1}{2}\left[\frac{t^2}{2} - t + \ln(t+1)\right]_0^1$ ***(A1)***

$= \frac{1}{4}$ (accept 0.25) ***A1***

[6 marks]

Examiners report

Again very few candidates gained full marks on this question. The most common approach was to begin by integrating by parts, which was done correctly, but very few candidates then knew how to integrate $\frac{t^2}{t+1}$. Those who began with a substitution often made more progress. Again a number of candidates were let down by their inability to simplify appropriately.

15a. *[4 marks]*

Markscheme

$\int x\sec^2 x\mathrm{d}x = x\tan x - \int 1 \times \tan x\mathrm{d}x$ ***M1A1***

$= x\tan x + \ln|\cos x|\ (+c)\ \ (= x\tan x - \ln|\sec x|\ (+c))$ ***M1A1***

[4 marks]

Examiners report

In part (a), a large number of candidates were able to use integration by parts correctly but were unable to use integration by substitution to then find the indefinite integral of tan *x*. In part (b), a large number of candidates attempted to solve the equation without direct use of a GDC's numerical solve command. Some candidates stated more than one solution for *m* and some specified *m* correct to two significant figures only.

15b. *[2 marks]*

Markscheme

attempting to solve an appropriate equation *eg* $m\tan m + \ln(\cos m) = 0.5$ ***(M1)***

m = 0.822 ***A1***

Note: Award ***A1*** if *m* = 0.822 is specified with other positive solutions.

[2 marks]

Examiners report

In part (a), a large number of candidates were able to use integration by parts correctly but were unable to use integration by substitution to then find the indefinite integral of tan *x*. In part (b), a large number of candidates attempted to solve the equation without direct use of a GDC's numerical solve command. Some candidates stated more than one solution for *m* and some specified *m* correct to two significant figures only.

16. *[5 marks]*

Markscheme

(a) $\int \frac{\sin\theta}{1-\cos\theta}\mathrm{d}\theta = \int \frac{(1-\cos\theta)'}{1-\cos\theta}\mathrm{d}\theta = \ln(1-\cos\theta) + C$ ***(M1)A1A1***

Note: Award ***A1*** for $\ln(1-\cos\theta)$ and ***A1*** for *C*.

(b) $\int_{\frac{\pi}{2}}^{a} \frac{\sin\theta}{1-\cos\theta}\mathrm{d}\theta = \frac{1}{2} \Rightarrow [\ln(1-\cos\theta)]_{\frac{\pi}{2}}^{a} = \frac{1}{2}$ ***M1***

$1 - \cos a = \mathrm{e}^{\frac{1}{2}} \Rightarrow a = \arccos(1-\sqrt{\mathrm{e}})$) or 2.28 ***A1 N2***

[5 marks]

Examiners report

Generally well answered, although many students did not include the constant of integration.

17. *[8 marks]*

Markscheme

$\int\sqrt{4-x^2}\mathrm{d}x$

$x = 2\sin\theta$

$\mathrm{d}x = 2\cos\theta\mathrm{d}\theta$ ***A1***

$= \int\sqrt{4-4\sin^2\theta} \times 2\cos\theta\mathrm{d}\theta$ ***M1A1***

$= \int 2\cos\theta \times 2\cos\theta\mathrm{d}\theta$

$= 4\int\cos^2\theta\mathrm{d}\theta$

now $\int\cos^2\theta\mathrm{d}\theta$

$= \int\left(\frac{1}{2}\cos 2\theta + \frac{1}{2}\right)\mathrm{d}\theta$ ***M1A1***

$= \left(\frac{\sin 2\theta}{4} + \frac{1}{2}\theta\right)$ ***A1***
so original integral
$= 2\sin 2\theta + 2\theta$
$= 2\sin\theta\cos\theta + 2\theta$
$= \left(2 \times \frac{x}{2} \times \frac{\sqrt{4-x^2}}{2}\right) + 2\arcsin\left(\frac{x}{2}\right)$
$= \frac{x\sqrt{4-x^2}}{2} + 2\arcsin\left(\frac{x}{2}\right) + C$ ***A1A1***
Note: Do not penalise omission of C.
$\left(A = \frac{1}{2},\ B = 2\right)$
[8 marks]
Examiners report
For many candidates this was an all or nothing question. Examiners were surprised at the number of candidates who were unable to change the variable in the integral using the given substitution. Another stumbling block, for some candidates, was a lack of care with the application of the trigonometric version of Pythagoras' Theorem to reduce the integrand to a multiple of $\cos^2\theta$. However, candidates who obtained the latter were generally successful in completing the question.

18a. *[5 marks]*
Markscheme
METHOD 1
sketch showing where the lines cross or zeros of $y = x(x+2)^6 - x$ ***(M1)***
$x = 0$ ***(A1)***
$x = -1$ and $x = -3$ ***(A1)***
the solution is $-3 < x < -1$ or $x > 0$ ***A1A1***
Note: Do not award either final ***A1*** mark if strict inequalities are not given.
METHOD 2
separating into two cases $x > 0$ and $x < 0$ ***(M1)***
if $x > 0$ then $(x+2)^6 > 1 \Rightarrow$ always true ***(M1)***
if $x < 0$ then $(x+2)^6 < 1 \Rightarrow -3 < x < -1$ ***(M1)***
so the solution is $-3 < x < -1$ or $x > 0$ ***A1A1***
Note: Do not award either final ***A1*** mark if strict inequalities are not given.
METHOD 3
$f(x) = x^7 + 12x^6 + 60x^5 + 160x^4 + 240x^3 + 192x^2 + 64x$ ***(A1)***
solutions to $x^7 + 12x^6 + 60x^5 + 160x^4 + 240x^3 + 192x^2 + 63x = 0$ are ***(M1)***
$x = 0,\ x = -1$ and $x = -3$ ***(A1)***
so the solution is $-3 < x < -1$ or $x > 0$ ***A1A1***
Note: Do not award either final ***A1*** mark if strict inequalities are not given.
METHOD 4
$f(x) = x$ when $x(x+2)^6 = x$
either $x = 0$ or $(x+2)^6 = 1$ ***(A1)***
if $(x+2)^6 = 1$ then $x + 2 = \pm 1$ so $x = -1$ or $x = -3$ ***(M1)(A1)***
the solution is $-3 < x < -1$ or $x > 0$ ***A1A1***
Note: Do not award either final ***A1*** mark if strict inequalities are not given.
[5 marks]
Examiners report
[N/A]

18b. *[5 marks]*
Markscheme
METHOD 1 (by substitution)
substituting $u = x + 2$ ***(M1)***
$\mathrm{d}u = \mathrm{d}x$
$\int (u-2)u^6\,\mathrm{d}u$ ***M1A1***
$= \frac{1}{8}u^8 - \frac{2}{7}u^7 (+c)$ ***(A1)***
$= \frac{1}{8}(x+2)^8 - \frac{2}{7}(x+2)^7 (+c)$ ***A1***

METHOD 2 (by parts)

$u = x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 1, \ \frac{\mathrm{d}v}{\mathrm{d}x} = (x+2)^6 \Rightarrow v = \frac{1}{7}(x+2)^7$ ***(M1)(A1)***

$\int x(x+2)^6 \mathrm{d}x = \frac{1}{7}x(x+2)^7 - \frac{1}{7}\int (x+2)^7 \mathrm{d}x$ ***M1***

$= \frac{1}{7}x(x+2)^7 - \frac{1}{56}(x+2)^8 (+c)$ ***A1A1***

METHOD 3 (by expansion)

$\int f(x)\mathrm{d}x = \int \left(x^7 + 12x^6 + 60x^5 + 160x^4 + 240x^3 + 192x^2 + 64x\right) \mathrm{d}x$ ***M1A1***

$= \frac{1}{8}x^8 + \frac{12}{7}x^7 + 10x^6 + 32x^5 + 60x^4 + 64x^3 + 32x^2 (+c)$ ***M1A2***

Note: Award ***M1A1*** if at least four terms are correct.

[5 marks]

Examiners report

[N/A]

19. *[7 marks]*

Markscheme

$x = a \sec\theta$

$\frac{\mathrm{d}x}{\mathrm{d}\theta} = a\sec\theta\tan\theta$ ***(A1)***

new limits:

$x = a\sqrt{2} \Rightarrow \theta = \frac{\pi}{4}$ and $x = 2a \Rightarrow \theta = \frac{\pi}{3}$ ***(A1)***

$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{a\sec\theta\tan\theta}{a^3\sec^3\theta\sqrt{a^2\sec^2\theta - a^2}} \mathrm{d}\theta$ ***M1***

$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\cos^2\theta}{a^3} \mathrm{d}\theta$ ***A1***

using $\cos^2\theta = \frac{1}{2}(\cos 2\theta + 1)$ ***M1***

$\frac{1}{2a^3}\left[\frac{1}{2}\sin 2\theta + \theta\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$ or equivalent ***A1***

$= \frac{1}{4a^3}\left(\frac{\sqrt{3}}{2} + \frac{2\pi}{3} - 1 - \frac{\pi}{2}\right)$ or equivalent ***A1***

$= \frac{1}{24a^3}\left(3\sqrt{3} + \pi - 6\right)$ ***AG***

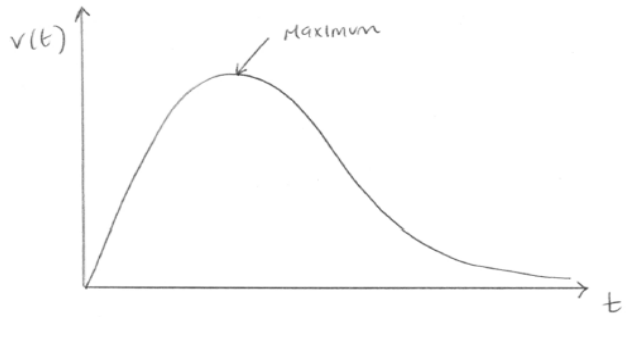
[7 marks]

Examiners report

[N/A]

20a. *[2 marks]*

Markscheme



(a) ***A1***

A1 for correct shape and correct domain

$(1.41, \ 0.0884) \ \left(\sqrt{2}, \ \frac{\sqrt{2}}{16}\right)$ ***A1***

[2 marks]

Examiners report

[N/A]

20b. *[4 marks]*

Markscheme

EITHER

$u = t^2$

$\frac{\mathrm{d}u}{\mathrm{d}t} = 2t$ ***A1***

OR

$t = u^{\frac{1}{2}}$

$\frac{\mathrm{d}t}{\mathrm{d}u} = \frac{1}{2}u^{-\frac{1}{2}}$ ***A1***

THEN

$\int \frac{t}{12+t^4}\,\mathrm{d}t = \frac{1}{2}\int \frac{\mathrm{d}u}{12+u^2}$ ***M1***

$= \frac{1}{2\sqrt{12}}\arctan\left(\frac{u}{\sqrt{12}}\right)(+c)$ ***M1***

$= \frac{1}{4\sqrt{3}}\arctan\left(\frac{t^2}{2\sqrt{3}}\right)(+c)$ or equivalent ***A1***

[4 marks]

Examiners report

[N/A]

20c. *[3 marks]*

Markscheme

$\int_0^6 \frac{t}{12+t^4}\,\mathrm{d}t$ ***(M1)***

$= \left[\frac{1}{4\sqrt{3}}\arctan\left(\frac{t^2}{2\sqrt{3}}\right)\right]_0^6$ ***M1***

$= \frac{1}{4\sqrt{3}}\left(\arctan\left(\frac{36}{2\sqrt{3}}\right)\right)\ \left(= \frac{1}{4\sqrt{3}}\left(\arctan\left(\frac{18}{\sqrt{3}}\right)\right)\right)\ (\mathrm{m})$ ***A1***

Note: Accept $\frac{\sqrt{3}}{12}\arctan\left(6\sqrt{3}\right)$ or equivalent.

[3 marks]

Examiners report

[N/A]

20d. *[3 marks]*

Markscheme

$\frac{\mathrm{d}v}{\mathrm{d}s} = \frac{1}{2\sqrt{s(1-s)}}$ ***(A1)***

$a = v\frac{\mathrm{d}v}{\mathrm{d}s}$

$a = \arcsin\left(\sqrt{s}\right) \times \frac{1}{2\sqrt{s(1-s)}}$ ***(M1)***

$a = \arcsin\left(\sqrt{0.1}\right) \times \frac{1}{2\sqrt{0.1\times0.9}}$

$a = 0.536\ (\mathrm{ms}^{-2})$ ***A1***

[3 marks]

Examiners report

[N/A]

21. *[7 marks]*

Markscheme

EITHER

$\frac{\mathrm{d}x}{\mathrm{d}u} = 2\sec^2 u$ ***A1***

$\int \frac{2\sec^2 u\mathrm{d}u}{4\tan^2 u\sqrt{4+4\tan^2 u}}$ ***(M1)***

$\int \frac{2\sec^2 u\mathrm{d}u}{4\tan^2 u\times 2\sec u}\ \left(= \int \frac{\mathrm{d}u}{4\sin^2 u\sqrt{\tan^2 u+1}}\ \text{or}\ = \int \frac{2\sec^2 u\mathrm{d}u}{4\tan^2 u\sqrt{4\sec^2 u}}\right)$ ***A1***

OR

$u = \arctan\frac{x}{2}$

$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{2}{x^2+4}$ ***A1***

$\int \frac{\sqrt{4\tan^2 u+4}\mathrm{d}u}{2\times 4\tan^2 u}$ ***(M1)***

$\int \frac{2\sec u\mathrm{d}u}{2\times 4\tan^2 u}$ ***A1***

THEN

$= \frac{1}{4}\int \frac{\sec u\mathrm{d}u}{\tan^2 u}$

$= \frac{1}{4}\int \operatorname{cosec} u \cot u \mathrm{d}u \ \left(= \frac{1}{4}\int \frac{\cos u}{\sin^2 u}\,\mathrm{d}u\right)$ ***A1***

$= -\frac{1}{4}\operatorname{cosec} u(+C) \ \left(= -\frac{1}{4\sin u}\,(+C)\right)$ ***A1***

use of either $u = \frac{x}{2}$ or an appropriate trigonometric identity ***M1***

either $\sin u = \frac{x}{\sqrt{x^2+4}}$ or $\operatorname{cosec} u = \frac{\sqrt{x^2+4}}{x}$ (or equivalent) ***A1***

$= \frac{-\sqrt{x^2+4}}{4x}\,(+C)$ ***AG***

[7 marks]

Examiners report

Most candidates found this a challenging question. A large majority of candidates were able to change variable from *x* to *u* but were not able to make any further progress.