Trig Functions, Equations & Identities May 2008-2014

1a. *[2 marks]*

Let $f(x) = \frac{\sin 3x}{\sin x} - \frac{\cos 3x}{\cos x}$.

For what values of *x* does $f(x)$ not exist?

Markscheme

$\cos x = 0,\ \sin x = 0$ ***(M1)***

$x = \frac{n\pi}{2}, n \in \mathbb{Z}$ ***A1***

1b. *[5 marks]*

Simplify the expression $\frac{\sin 3x}{\sin x} - \frac{\cos 3x}{\cos x}$.

Markscheme

EITHER

$\frac{\sin 3x \cos x - \cos 3x \sin x}{\sin x \cos x}$ ***M1 A1***

$= \frac{\sin(3x - x)}{\frac{1}{2}\sin 2x}$ ***A1 A1***

$= 2$ ***A1***

OR

$\frac{\sin 2x \cos x + \cos 2x \sin x}{\sin x} - \frac{\cos 2x \cos x - \sin 2x \sin x}{\cos x}$ ***M1***

$= \frac{2 \sin x \cos^2 x + 2\cos^2 x \sin x - \sin x}{\sin x} - \frac{2\cos^3 x - \cos x - \sin^2 x \cos x}{\cos x}$ ***A1 A1***

$= 4\cos^2 x - 1 - 2\cos^2 x + 1 + 2\sin^2 x$ ***A1*** $= 2\cos^2 x + 2\sin^2 x$

$= 2$ ***A1***

[5 marks]

2a. *[3 marks]*

In the triangle ABC, $\mathrm{A\hat{B}C} = 90°$, $\mathrm{AC} = \sqrt{2}$ and AB = BC + 1.

Show that cos $\hat{A} - \sin \hat{A} = \frac{1}{\sqrt{2}}$.

Markscheme

$\cos \hat{A} = \frac{\mathrm{BA}}{\sqrt{2}}$ ***A1***

$\sin \hat{A} = \frac{\mathrm{BC}}{\sqrt{2}}$ ***A1***

$\cos \hat{A} - \sin \hat{A} = \frac{\mathrm{BA} - \mathrm{BC}}{\sqrt{2}}$ ***R1***

$= \frac{1}{\sqrt{2}}$ ***AG***

[3 marks]

2b. *[8 marks]*

By squaring both sides of the equation in part (a), solve the equation to find the angles in the triangle.

Markscheme

$\cos^2 \hat{A} - 2\cos \hat{A} \sin \hat{A} + \sin^2 \hat{A} = \frac{1}{2}$ ***M1A1***

$1 - 2\sin \hat{A} \cos \hat{A} = \frac{1}{2}$ ***M1A1***

$\sin 2\hat{A} = \frac{1}{2}$ ***M1***

$2\hat{A} = 30°$ ***A1***

angles in the triangle are 15° and 75° ***A1A1***

Note: Accept answers in radians.

[8 marks]

2c. *[6 marks]*

Apply Pythagoras' theorem in the triangle ABC to find BC, and hence show that $\sin \hat{A} = \frac{\sqrt{6} - \sqrt{2}}{4}$.

Markscheme

$\mathrm{BC}^2 + (\mathrm{BC} + 1)^2 = 2$ ***M1A1***

$2\mathrm{BC}^2 + 2\mathrm{BC} - 1 = 0$ ***A1***

$\mathrm{BC} = \frac{-2 + \sqrt{12}}{4} \left(= \frac{\sqrt{3} - 1}{2}\right)$ ***M1A1***

$\sin \hat{A} = \frac{\mathrm{BC}}{\sqrt{2}} = \frac{\sqrt{3} - 1}{2\sqrt{2}}$ ***A1***

$= \frac{\sqrt{6}-\sqrt{2}}{4} AG$

[6 marks]

2d. *[4 marks]*

Hence, or otherwise, calculate the length of the perpendicular from B to [AC].

Markscheme

EITHER

$h = \text{AB} \sin \hat{A}$ ***M1***

$= (\text{BC} + 1) \sin \hat{A}$ ***A1***

$= \frac{\sqrt{3}+1}{2} \times \frac{\sqrt{6}-\sqrt{2}}{4} = \frac{\sqrt{2}}{4}$ ***M1A1***

OR

$\frac{1}{2} AB.BC = \frac{1}{2} AC.h$ ***M1***

$\frac{\sqrt{3}-1}{2} \cdot \frac{\sqrt{3}+1}{2} = \sqrt{2}h$ ***A1***

$\frac{2}{4} = \sqrt{2}h$ ***M1***

$h = \frac{1}{2\sqrt{2}}$ ***A1***

[4 marks]

3a. *[1 mark]*

The function $f(x) = 3\sin x + 4\cos x$ is defined for $0 < x < 2\pi$.

Write down the coordinates of the minimum point on the graph of *f* .

Markscheme

$(3.79, -5)$ ***A1***

[1 mark]

3b. *[2 marks]*

The points $\text{P}(p,\ 3)$ and $\text{Q}(q,\ 3), q > p$, lie on the graph of $y = f(x)$.

Find *p* and *q* .

Markscheme

$p = 1.57$ or $\frac{\pi}{2}$, $q = 6.00$ ***A1A1***

[2 marks]

3c. *[4 marks]*

Find the coordinates of the point, on $y = f(x)$, where the gradient of the graph is 3.

Markscheme

$f'(x) = 3\cos x - 4\sin x$ ***(M1)(A1)***

$3\cos x - 4\sin x = 3 \Rightarrow x = 4.43...$ ***(A1)***

$(y = -4)$ ***A1***

Coordinates are $(4.43, -4)$

[4 marks]

3d. *[7 marks]*

Find the coordinates of the point of intersection of the normals to the graph at the points P and Q.

Markscheme

$m_{\text{normal}} = \frac{1}{m_{\text{tangent}}}$ ***(M1)***

gradient at P is -4 so gradient of normal at P is $\frac{1}{4}$ ***(A1)***

gradient at Q is 4 so gradient of normal at Q is $-\frac{1}{4}$ ***(A1)***

equation of normal at P is $y - 3 = \frac{1}{4}(x - 1.570...)$ (or $y = 0.25x + 2.60...$) ***(M1)***

$y - 3 = \frac{1}{4}(x - 5.999...)$ (or $y = -0.25x + \underbrace{4.499...}$)

equation of normal at Q is ***(M1)***

Note: Award the previous two ***M1*** even if the gradients are incorrect in $y - b = m(x - a)$ where (a, b) are coordinates of P and Q (or in $y = mx + c$ with *c* determined using coordinates of P and Q.

intersect at $(3.79,\ 3.55)$ ***A1A1***

Note: Award ***N2*** for 3.79 without other working.

[7 marks]

5. *[6 marks]*

Show that $\frac{\cos A + \sin A}{\cos A - \sin A} = \sec 2A + \tan 2A$.

Markscheme

METHOD 1

$\frac{\cos A+\sin A}{\cos A-\sin A}=\sec 2A+\tan 2A$

consider right hand side

$\sec 2A+\tan 2A=\frac{1}{\cos 2A}+\frac{\sin 2A}{\cos 2A}$ ***M1A1***

$=\frac{\cos^2 A+2\sin A\cos A+\sin^2 A}{\cos^2 A-\sin^2 A}$ ***A1A1***

Note: Award ***A1*** for recognizing the need for single angles and ***A1*** for recognizing $\cos^2 A+\sin^2 A=1$.

$=\frac{(\cos A+\sin A)^2}{(\cos A+\sin A)(\cos A-\sin A)}$ ***M1A1***

$=\frac{\cos A+\sin A}{\cos A-\sin A}$ ***AG***

METHOD 2

$\frac{\cos A+\sin A}{\cos A-\sin A}=\frac{(\cos A+\sin A)^2}{(\cos A+\sin A)(\cos A-\sin A)}$ ***M1A1***

$=\frac{\cos^2 A+2\sin A\cos A+\sin^2 A}{\cos^2 A-\sin^2 A}$ ***A1A1***

Note: Award ***A1*** for correct numerator and ***A1*** for correct denominator.

$=\frac{1+\sin 2A}{\cos 2A}$ ***M1A1***

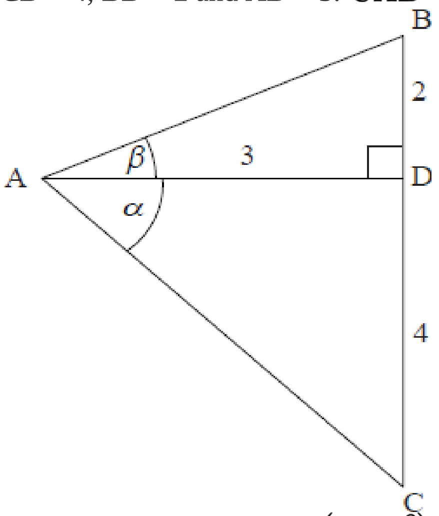
$=\sec 2A+\tan 2A$ ***AG***

[6 marks]

6. *[6 marks]*

In the diagram below, AD is perpendicular to BC.

CD = 4, BD = 2 and AD = 3. $\mathrm{C\hat{A}D}=\alpha$ and $\mathrm{B\hat{A}D}=\beta$.



Find the exact value of $\cos(\alpha-\beta)$.

Markscheme

METHOD 1

$\mathrm{AC}=5$ and $\mathrm{AB}=\sqrt{13}$ (may be seen on diagram) ***(A1)***

$\cos\alpha=\frac{3}{5}$ **and** $\sin\alpha=\frac{4}{5}$ ***(A1)***

$\cos\beta=\frac{3}{\sqrt{13}}$ **and** $\sin\beta=\frac{2}{\sqrt{13}}$ ***(A1)***

Note: If only the two cosines are correctly given award ***(A1)(A1)(A0)***.

Use of $\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$ ***(M1)***

$=\frac{3}{5}\times\frac{3}{\sqrt{13}}+\frac{4}{5}\times\frac{2}{\sqrt{13}}$ (substituting) ***M1***

$=\frac{17}{5\sqrt{13}}\left(=\frac{17\sqrt{13}}{65}\right)$ ***A1 N1***

[6 marks]

METHOD 2

$\mathrm{AC}=5$ amd $\mathrm{AB}=\sqrt{13}$ (may be seen on diagram) ***(A1)***

Use of $\cos(\alpha+\beta)=\frac{\mathrm{AC}^2+\mathrm{AB}^2-\mathrm{BC}^2}{2(\mathrm{AC})(\mathrm{AB})}$ ***(M1)***

$=\frac{25+13-36}{2\times5\times\sqrt{13}}\quad\left(=\frac{1}{5\sqrt{13}}\right)$ ***A1***

Use of $\cos(\alpha+\beta)+\cos(\alpha-\beta)=2\cos\alpha\cos\beta$ ***(M1)***

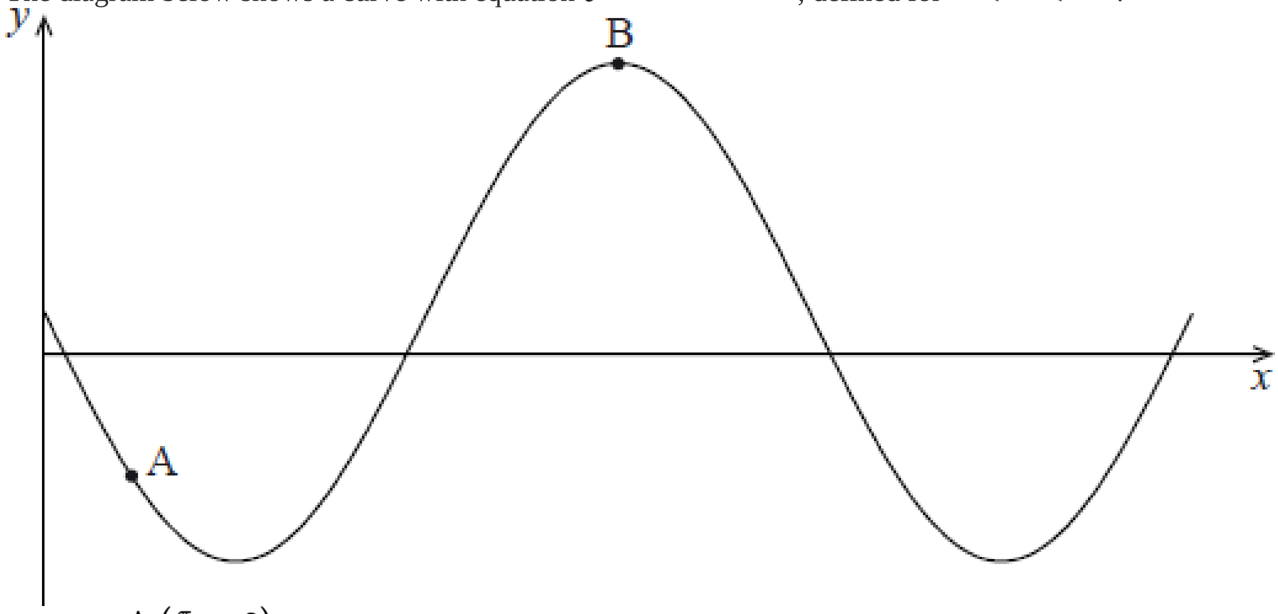
$\cos\alpha=\frac{3}{5}$ and $\cos\beta=\frac{3}{\sqrt{13}}$ ***(A1)***

$\cos(\alpha-\beta)=\frac{17}{5\sqrt{13}} \quad \left(=2\times\frac{3}{5}\times\frac{3}{\sqrt{13}}-\frac{1}{5\sqrt{13}}\right) \quad \left(=\frac{17\sqrt{13}}{65}\right)$ ***A1 N1***

[6 marks]

9. *[5 marks]*

The diagram below shows a curve with equation $y=1+k\sin x$, defined for $0\leqslant x\leqslant 3\pi$.



The point $\mathrm{A}\left(\frac{\pi}{6},-2\right)$ lies on the curve and $\mathrm{B}(a,\ b)$ is the maximum point.

(a) Show that $k=-6$.

(b) Hence, find the values of a and b.

Markscheme

(a) $-2=1+k\sin\left(\frac{\pi}{6}\right)$ ***M1***

$-3=\frac{1}{2}k$ ***A1***

$k=-6$ ***AG N0***

(b) **METHOD 1**

maximum $\Rightarrow \sin x=-1$ ***M1***

$a=\frac{3\pi}{2}$ ***A1***

$b=1-6(-1)$

$=7$ ***A1 N2***

METHOD 2

$y'=0$ ***M1***

$k\cos x=0\Rightarrow x=\frac{\pi}{2},\ \frac{3\pi}{2},\ \ldots$

$a=\frac{3\pi}{2}$ ***A1***

$b=1-6(-1)$

$=7$ ***A1 N2***

Note: Award ***A1A1*** for $\left(\frac{3\pi}{2},\ 7\right)$.

[5 marks]

10. *[5 marks]*

(a) Show that $\arctan\left(\frac{1}{2}\right)+\arctan\left(\frac{1}{3}\right)=\frac{\pi}{4}$.

(b) Hence, or otherwise, find the value of $\arctan(2)+\arctan(3)$.

Markscheme

(a) **METHOD 1**

let $x=\arctan\frac{1}{2}\Rightarrow\tan x=\frac{1}{2}$ and $y=\arctan\frac{1}{3}\Rightarrow\tan y=\frac{1}{3}$

$\tan(x+y)=\frac{\tan x+\tan y}{1-\tan x\tan y}=\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\times\frac{1}{3}}=1$ ***M1***

so, $x+y=\arctan 1=\frac{\pi}{4}$ ***A1AG***

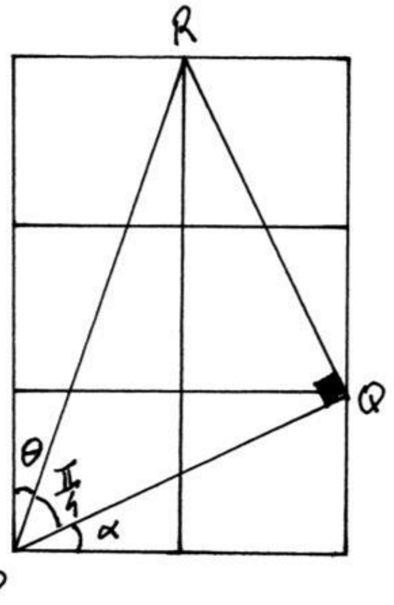
METHOD 2

for $x,\ y > 0$, $\arctan x + \arctan y = \arctan\left(\frac{x+y}{1-xy}\right)$ if $xy < 1$ ***M1***

so, $\arctan\frac{1}{2} + \arctan\frac{1}{3} = \arctan\left(\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\times\frac{1}{3}}\right) = \frac{\pi}{4}$ ***A1AG***

METHOD 3

an appropriate sketch ***M1***



e.g.

correct reasoning leading to $\frac{\pi}{4}$ ***R1AG***

(b) **METHOD 1**

$\arctan(2) + \arctan(3) = \frac{\pi}{2} - \arctan\left(\frac{1}{2}\right) + \frac{\pi}{2} - \arctan\left(\frac{1}{3}\right)$ ***(M1)***

$= \pi - \left(\arctan\left(\frac{1}{2}\right) + \arctan\left(\frac{1}{3}\right)\right)$ ***(A1)***

Note: Only one of the previous two marks may be implied.

$= \pi - \frac{\pi}{4} = \frac{3\pi}{4}$ ***A1 N1***

METHOD 2

let $x = \arctan 2 \Rightarrow \tan x = 2$ and $y = \arctan 3 \Rightarrow \tan y = 3$

$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y} = \frac{2+3}{1-2\times 3} = -1$ ***(M1)***

as $\frac{\pi}{4} < x < \frac{\pi}{2}$ $\left(\text{accept } 0 < x < \frac{\pi}{2}\right)$

and $\frac{\pi}{4} < y < \frac{\pi}{2}$ $\left(\text{accept } 0 < y < \frac{\pi}{2}\right)$

$\frac{\pi}{2} < x + y < \pi$ $\left(\text{accept } 0 < x + y < \pi\right)$ ***(R1)***

Note: Only one of the previous two marks may be implied.

so, $x + y = \frac{3\pi}{4}$ ***A1 N1***

METHOD 3

for $x,\ y > 0$, $\arctan x + \arctan y = \arctan\left(\frac{x+y}{1-xy}\right) + \pi$ if $xy > 1$ ***(M1)***

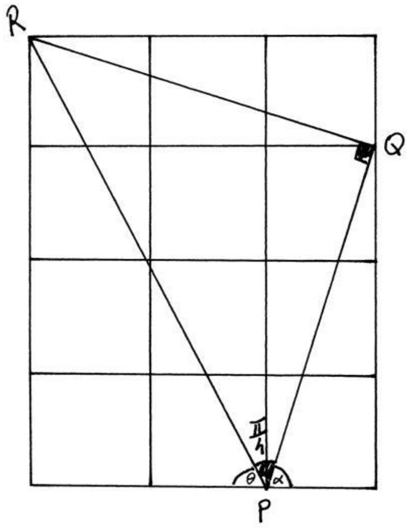
so, $\arctan 2 + \arctan 3 = \arctan\left(\frac{2+3}{1-2\times 3}\right) + \pi$ ***(A1)***

Note: Only one of the previous two marks may be implied.

$= \frac{3\pi}{4}$ ***A1 N1***

METHOD 4

an appropriate sketch ***M1***



e.g.

correct reasoning leading to $\frac{3\pi}{4}$ ***R1A1***

[5 marks]

12. *[20 marks]*

(a) Show that $\sin 2nx = \sin((2n+1)x)\cos x - \cos((2n+1)x)\sin x$.

(b) **Hence** prove, by induction, that

$$\cos x + \cos 3x + \cos 5x + \ldots + \cos((2n-1)x) = \frac{\sin 2nx}{2\sin x},$$

for all $n \in \mathbb{Z}^+,\ \sin x \neq 0$.

(c) Solve the equation $\cos x + \cos 3x = \frac{1}{2},\ 0 < x < \pi$.

Markscheme

(a) $\sin(2n+1)x\cos x - \cos(2n+1)x\sin x = \sin(2n+1)x - x$ ***M1A1***

$= \sin 2nx$ ***AG***

[2 marks]

(b) if *n* = 1 ***M1***

$\text{LHS} = \cos x$

$\text{RHS} = \frac{\sin 2x}{2\sin x} = \frac{2\sin x\cos x}{2\sin x} = \cos x$ ***M1***

so LHS = RHS and the statement is true for *n* = 1 ***R1***

assume true for *n* = *k* ***M1***

Note: Only award ***M1*** if the word **true** appears.

Do **not** award ***M1*** for ‘let *n* = *k*’ only.

Subsequent marks are independent of this ***M1***.

so $\cos x + \cos 3x + \cos 5x + \ldots + \cos(2k-1)x = \frac{\sin 2kx}{2\sin x}$

if *n* = *k* + 1 then

$\cos x + \cos 3x + \cos 5x + \ldots + \cos(2k-1)x + \cos(2k+1)x$ ***M1***

$= \frac{\sin 2kx}{2\sin x} + \cos(2k+1)x$ ***A1***

$= \frac{\sin 2kx + 2\cos(2k+1)x\sin x}{2\sin x}$ ***M1***

$= \frac{\sin(2k+1)x\cos x - \cos(2k+1)x\sin x + 2\cos(2k+1)x\sin x}{2\sin x}$ ***M1***

$= \frac{\sin(2k+1)x\cos x + \cos(2k+1)x\sin x}{2\sin x}$ ***A1***

$= \frac{\sin(2k+2)x}{2\sin x}$ ***M1***

$= \frac{\sin 2(k+1)x}{2\sin x}$ ***A1***

so if true for *n* = *k*, then also true for *n* = *k* + *1*

as true for *n* = 1 then true for all $n \in \mathbb{Z}^+$ ***R1***

Note: Final ***R1*** is independent of previous work.

[12 marks]

(c) $\frac{\sin 4x}{2\sin x} = \frac{1}{2}$ ***M1A1***

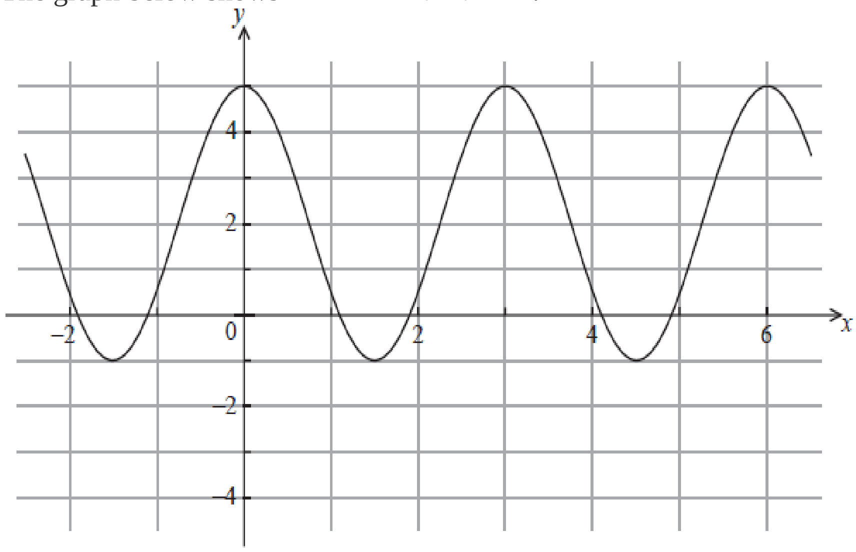
$\sin 4x = \sin x$

$4x = x \Rightarrow x = 0$ but this is impossible

$4x = \pi - x \Rightarrow x = \frac{\pi}{5}$ ***A1***
$4x = 2\pi + x \Rightarrow x = \frac{2\pi}{3}$ ***A1***
$4x = 3\pi - x \Rightarrow x = \frac{3\pi}{5}$ ***A1***
for not including any answers outside the domain ***R1***
Note: Award the first ***M1A1*** for correctly obtaining $8\cos^3 x - 4\cos x - 1 = 0$ or equivalent and subsequent marks as appropriate including the answers $\left(-\frac{1}{2}, \frac{1 \pm \sqrt{5}}{4}\right)$.
[6 marks]
Total [20 marks]

13. *[4 marks]*

The graph below shows $y = a\cos(bx) + c$.



Find the value of *a*, the value of *b* and the value of *c*.

Markscheme

$a = 3$ ***A1***
$c = 2$ ***A1***
period $= \frac{2\pi}{b} = 3$ ***(M1)***
$b = \frac{2\pi}{3}\ (= 2.09)$ ***A1***
[4 marks]

14. *[6 marks]*

If *x* satisfies the equation $\sin\left(x + \frac{\pi}{3}\right) = 2\sin x \sin\left(\frac{\pi}{3}\right)$, show that $11\tan x = a + b\sqrt{3}$, where $a, b \in \mathbb{Z}^+$.

Markscheme

$\sin\left(x + \frac{\pi}{3}\right) = \sin x \cos\left(\frac{\pi}{3}\right) + \cos x \sin\left(\frac{\pi}{3}\right)$ ***(M1)***
$\sin x \cos\left(\frac{\pi}{3}\right) + \cos x \sin\left(\frac{\pi}{3}\right) = 2\sin x \sin\left(\frac{\pi}{3}\right)$
$\frac{1}{2}\sin x + \frac{\sqrt{3}}{2}\cos x = 2 \times \frac{\sqrt{3}}{2}\sin x$ ***A1***
dividing by $\cos x$ and rearranging ***M1***
$\tan x = \frac{\sqrt{3}}{2\sqrt{3} - 1}$ ***A1***
rationalizing the denominator ***M1***
$11\tan x = 6 + \sqrt{3}$ ***A1***
[6 marks]

15a. *[3 marks]*

Given that $\arctan\left(\frac{1}{5}\right) + \arctan\left(\frac{1}{8}\right) = \arctan\left(\frac{1}{p}\right)$, where $p \in \mathbb{Z}^+$, find *p*.

Markscheme

attempt at use of $\tan(A + B) = \frac{\tan(A) + \tan(B)}{1 - \tan(A)\tan(B)}$ ***M1***
$\frac{1}{p} = \frac{\frac{1}{5} + \frac{1}{8}}{1 - \frac{1}{5} \times \frac{1}{8}}\ \left(= \frac{1}{3}\right)$ ***A1***
$p = 3$ ***A1***

Note: the value of *p* needs to be stated for the final mark.

[3 marks]

15b. *[3 marks]*

Hence find the value of $\arctan\left(\frac{1}{2}\right)+\arctan\left(\frac{1}{5}\right)+\arctan\left(\frac{1}{8}\right)$.

Markscheme

$\tan\left(\arctan\left(\frac{1}{2}\right)+\arctan\left(\frac{1}{5}\right)+\arctan\left(\frac{1}{8}\right)\right)=\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\times\frac{1}{3}}=1$ ***M1A1***

$\arctan\left(\frac{1}{2}\right)+\arctan\left(\frac{1}{5}\right)+\arctan\left(\frac{1}{8}\right)=\frac{\pi}{4}$ ***A1***

[3 marks]

16a. *[3 marks]*

Solve the equation $3\cos^2 x-8\cos x+4=0$, where $0\leqslant x\leqslant 180^\circ$, expressing your answer(s) to the nearest degree.

Markscheme

attempting to solve for $\cos x$ or for *u* where $u=\cos x$ or for *x* graphically. ***(M1)***

EITHER

$\cos x=\frac{2}{3}\ (\text{and } 2)$ ***(A1)***

OR

$x=48.1897\ldots^\circ$ ***(A1)***

THEN

$x=48^\circ$ ***A1***

Note: Award ***(M1)(A1)A0*** for $x=48^\circ,\ 132^\circ$.

Note: Award ***(M1)(A1)A0*** for 0.841 radians.

[3 marks]

16b. *[3 marks]*

Find the exact values of $\sec x$ satisfying the equation $3\sec^4 x-8\sec^2 x+4=0$.

Markscheme

attempting to solve for $\sec x$ or for v where $v=\sec x$. ***(M1)***

$\sec x=\pm\sqrt{2}\ \left(\text{and } \pm\sqrt{\frac{2}{3}}\right)$ ***(A1)***

$\sec x=\pm\sqrt{2}$ ***A1***

[3 marks]

17a. *[2 marks]*

Show that $\frac{\sin 2\theta}{1+\cos 2\theta}=\tan\theta$.

Markscheme

$\frac{\sin 2\theta}{1+\cos 2\theta}=\frac{2\sin\theta\cos\theta}{1+2\cos^2\theta-1}$ ***M1***

Note: Award ***M1*** for use of double angle formulae.

$=\frac{2\sin\theta\cos\theta}{2\cos^2\theta}$ ***A1***

$=\frac{\sin\theta}{\cos\theta}$

$=\tan\theta$ ***AG***

[2 marks]

17b. *[3 marks]*

Hence find the value of $\cot\frac{\pi}{8}$ in the form $a+b\sqrt{2}$, where $a,b\in\mathbb{Z}$.

Markscheme

$\tan\frac{\pi}{8}=\frac{\sin\frac{\pi}{4}}{1+\cos\frac{\pi}{4}}$ ***(M1)***

$\cot\frac{\pi}{8}=\frac{1+\cos\frac{\pi}{4}}{\sin\frac{\pi}{4}}$ ***M1***

$=\frac{1+\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}}$

$=1+\sqrt{2}$ ***A1***

[3 marks]

19a. *[2 marks]*

Use the identity $\cos 2\theta=2\cos^2\theta-1$ to prove that $\cos\frac{1}{2}x=\sqrt{\frac{1+\cos x}{2}},\ 0\leqslant x\leqslant\pi$.

Markscheme

$\cos x = 2\cos^2 \frac{1}{2}x - 1$

$\cos \frac{1}{2}x = \pm\sqrt{\frac{1+\cos x}{2}}$ ***M1***

positive as $0 \leqslant x \leqslant \pi$ ***R1***

$\cos \frac{1}{2}x = \sqrt{\frac{1+\cos x}{2}}$ ***AG***

[2 marks]

19b. *[2 marks]*

Find a similar expression for $\sin \frac{1}{2}x,\ 0 \leqslant x \leqslant \pi$.

Markscheme

$\cos 2\theta = 1 - 2\sin^2\theta$ ***(M1)***

$\sin \frac{1}{2}x = \sqrt{\frac{1-\cos x}{2}}$ ***A1***

[2 marks]

19c. *[4 marks]*

Hence find the value of $\int_0^{\frac{\pi}{2}} \left(\sqrt{1+\cos x} + \sqrt{1-\cos x}\right) \mathrm{d}x$.

Markscheme

$\sqrt{2}\int_0^{\frac{\pi}{2}} \cos \frac{1}{2}x + \sin \frac{1}{2}x\mathrm{d}x$ ***A1***

$= \sqrt{2}\left[2\sin \frac{1}{2}x - 2\cos \frac{1}{2}x\right]_0^{\frac{\pi}{2}}$ ***A1***

$= \sqrt{2}(0) - \sqrt{2}(0-2)$ ***A1***

$= 2\sqrt{2}$ ***A1***

[4 marks]

20. *[6 marks]*

Given that $\sin x + \cos x = \frac{2}{3}$, find $\cos 4x$.

Markscheme

$\sin^2 x + \cos^2 x + 2\sin x\cos x = \frac{4}{9}$ ***(M1)(A1)***

using $\sin^2 x + \cos^2 x = 1$ ***(M1)***

$2\sin x\cos x = -\frac{5}{9}$

using $2\sin x\cos x = \sin 2x$ ***(M1)***

$\sin 2x = -\frac{5}{9}$

$\cos 4x = 1 - 2\sin^2 2x$ ***M1***

Note: Award this ***M1*** for decomposition of cos 4*x* using double angle formula anywhere in the solution.

$= 1 - 2 \times \frac{25}{81}$

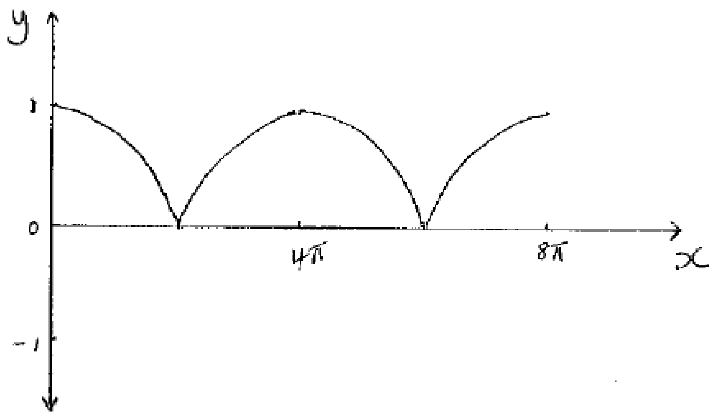
$= \frac{31}{81}$ ***A1***

[6 marks]

22a. *[2 marks]*

Sketch the graph of $y = \left|\cos\left(\frac{x}{4}\right)\right|$ for $0 \leqslant x \leqslant 8\pi$.

Markscheme



A1A1

Note: Award ***A1*** for correct shape and ***A1*** for correct domain and range.

[2 marks]

22b. *[3 marks]*

Solve $\left|\cos\left(\frac{x}{4}\right)\right| = \frac{1}{2}$ for $0 \leqslant x \leqslant 8\pi$.

Markscheme

$\left|\cos\left(\frac{x}{4}\right)\right| = \frac{1}{2}$

$x = \frac{4\pi}{3}$ ***A1***

attempting to find any other solutions ***M1***

Note: Award ***(M1)*** if at least one of the other solutions is correct (in radians or degrees) or clear use of symmetry is seen.

$x = 8\pi - \frac{4\pi}{3} = \frac{20\pi}{3}$

$x = 4\pi - \frac{4\pi}{3} = \frac{8\pi}{3}$

$x = 4\pi + \frac{4\pi}{3} = \frac{16\pi}{3}$ ***A1***

Note: Award ***A1*** for all other three solutions correct and no extra solutions.

Note: If working in degrees, then max ***A0M1A0***.

[3 marks]

23. *[7 marks]*

The first three terms of a geometric sequence are $\sin x,\ \sin 2x$ and $4\sin x\cos^2 x,\ -\frac{\pi}{2} < x < \frac{\pi}{2}$.

(a) Find the common ratio *r*.

(b) Find the set of values of *x* for which the geometric series $\sin x + \sin 2x + 4\sin x\cos^2 x + \ldots$ converges.

Consider $x = \arccos\left(\frac{1}{4}\right),\ x > 0$.

(c) Show that the sum to infinity of this series is $\frac{\sqrt{15}}{2}$.

Markscheme

(a) $\sin x,\ \sin 2x \text{ and } 4\sin x\cos^2 x$

$r = \frac{2\sin x\cos x}{\sin x} = 2\cos x$ **A1**

Note: Accept $\frac{\sin 2x}{\sin x}$.

[1 mark]

(b) **EITHER**

$|r| < 1 \Rightarrow |2\cos x| < 1$ ***M1***

OR

$-1 < r < 1 \Rightarrow -1 < 2\cos x < 1$ ***M1***

THEN

$0 < \cos x < \frac{1}{2} \text{ for } -\frac{\pi}{2} < x < \frac{\pi}{2}$

$-\frac{\pi}{2} < x < -\frac{\pi}{3} \text{ or } \frac{\pi}{3} < x < \frac{\pi}{2}$ ***A1A1***

[3 marks]

(c) $S_\infty = \frac{\sin x}{1-2\cos x}$ ***M1***

$S_\infty = \frac{\sin\left(\arccos\left(\frac{1}{4}\right)\right)}{1-2\cos\left(\arccos\left(\frac{1}{4}\right)\right)}$

$= \frac{\frac{\sqrt{15}}{4}}{\frac{1}{2}}$ ***A1A1***

Note: Award ***A1*** for correct numerator and ***A1*** for correct denominator.

$= \frac{\sqrt{15}}{2}$ ***AG***

[3 marks]

Total [7 marks]

24a. *[2 marks]*

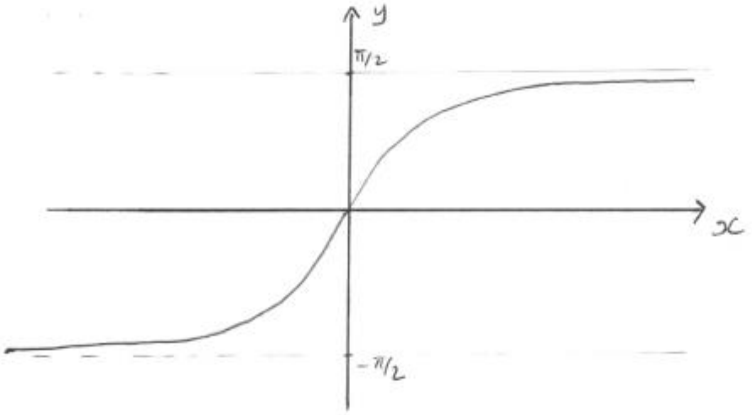
Consider the following functions:

$h(x) = \arctan(x),\ x \in \mathbb{R}$

$g(x) = \frac{1}{x}, x \in \mathbb{R},\ x \neq 0$

Sketch the graph of $y = h(x)$.

Markscheme



A1A1

Note: ***A1*** for correct shape, ***A1*** for asymptotic behaviour at $y = \pm \frac{\pi}{2}$.

[2 marks]

24b. *[2 marks]*

Find an expression for the composite function $h \circ g(x)$ and state its domain.

Markscheme

$h \circ g(x) = \arctan\left(\frac{1}{x}\right)$ ***A1***

domain of $h \circ g$ is equal to the domain of $g : x \in \circ,\ x \neq 0$ ***A1***

[2 marks]

24c. *[7 marks]*

Given that $f(x) = h(x) + h \circ g(x)$,

(i) find $f'(x)$ in simplified form;

(ii) show that $f(x) = \frac{\pi}{2}$ for $x > 0$.

Markscheme

(i) $f(x) = \arctan(x) + \arctan\left(\frac{1}{x}\right)$

$f'(x) = \frac{1}{1+x^2} + \frac{1}{1+\frac{1}{x^2}} \times -\frac{1}{x^2}$ ***M1A1***

$f'(x) = \frac{1}{1+x^2} + \frac{-\frac{1}{x^2}}{\frac{x^2+1}{x^2}}$ ***(A1)***

$= \frac{1}{1+x^2} - \frac{1}{1+x^2}$

$= 0$ ***A1***

(ii) **METHOD 1**

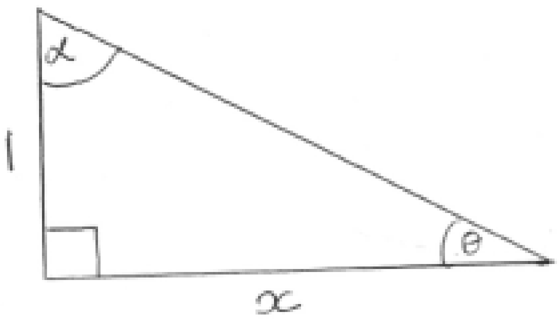
f is a constant ***R1***

when $x > 0$

$f(1) = \frac{\pi}{4} + \frac{\pi}{4}$ ***M1A1***

$= \frac{\pi}{2}$ ***AG***

METHOD 2



from diagram

$\theta = \arctan \frac{1}{x}$ ***A1***

$\alpha = \arctan x$ ***A1***

$\theta + \alpha = \frac{\pi}{2}$ ***R1***

hence $f(x) = \frac{\pi}{2}$ ***AG***

METHOD 3

$\tan(f(x)) = \tan\left(\arctan(x) + \arctan\left(\frac{1}{x}\right)\right)$ ***M1***

$= \frac{x+\frac{1}{x}}{1-x\left(\frac{1}{x}\right)}$ ***A1***

denominator = 0, so $f(x) = \frac{\pi}{2}$ (for $x > 0$) ***R1***

[7 marks]

24d. *[3 marks]*

Nigel states that f is an odd function and Tom argues that f is an even function.

(i) State who is correct and justify your answer.

(ii) Hence find the value of $f(x)$ for $x < 0$.

Markscheme

(i) Nigel is correct. ***A1***

METHOD 1

$\arctan(x)$ is an odd function and $\frac{1}{x}$ is an odd function

composition of two odd functions is an odd function and sum of two odd functions is an odd function ***R1***

METHOD 2

$f(-x) = \arctan(-x) + \arctan\left(-\frac{1}{x}\right) = -\arctan(x) - \arctan\left(\frac{1}{x}\right) = -f(x)$

therefore *f* is an odd function. ***R1***

(ii) $f(x) = -\frac{\pi}{2}$ ***A1***

[3 marks]